Stability of Petrov-Galerkin discretizations: Application to the space-time weak formulation for parabolic equations

Talk No. 2

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Overview

The Space-Time Weak Formulation of parabolic equations

Choice of spaces and verification of the first assumption

Discretisation and Verification of the second assumption

The Space-Time Weak Formulation of parabolic equations

The parabolic equation

Let V and H be Hilbert spaces with embedding with dense embedding $V \hookrightarrow H$. By identifying H with H', we obtain the Gelfand triple

$$V \hookrightarrow H \hookrightarrow V'.$$

Now let for any $t \in [0, T]$

$$A(t) : V \to V'$$

be a linear operator. We assume that

- for all $v, w \in V$ the mapping $t \mapsto \langle A(t)v, w\rangle$ is measurable.
- for $t \in [0, 1]$ the operator $A(t)$ is bounded fulfils

$$\langle A(t)v, v\rangle + \lambda \|v\|_H^2 \geq \alpha \|v\|_V^2 \text{ for } v \in V.$$

Weak formulation of the parabolic problem

Given a right hand side $f \in L^2(0, T; V')$ and an initial condition $u_0 \in H$ find a solution $u(t) \in V$ such that

$$\begin{aligned} \frac{du}{dt}(t) + A(t)u(t) &= f(t) && \text{in } V' \\ u(0) &= u_0 && \text{in } H. \end{aligned}$$

Space-time weak formulation of the parabolic problem

Multiplying with a testfunction exin $H^1(0, T; V)$ and applying partial integration yields the space-time formulation:
Find

$$u \in X = L^2(0, T; V)$$

such that for all

$$q \in Y = L^2(0, T; V) \cap H^1_{\{T\}}(0, T; V')$$

we have

$$-\int_0^T \left\langle u(t), \frac{\mathrm{d}q(t)}{\mathrm{d}t} \right\rangle_{V \times V'} \mathrm{d}t + \int_0^T \langle A(t)u(t), q(t) \rangle_{V' \times V} \, \mathrm{d}t$$
$$= \int_0^T \langle f(t), q(t) \rangle_{V' \times V} \, \mathrm{d}t + (u_0, q(0))_H$$

or equivalently

$$\langle Bu, q \rangle_{Y' \times Y} = \ell(q).$$

Norms on X and Y

On the spaces X and Y we define norms by

$$\|u\|_X^2 = \|u\|_{L^2(0,T;V)}^2$$

and

$$\|q\|_Y^2 = \|q\|_{L^2(0,T;V)}^2 + \|q\|_{L^2(0,T;V')}^2 + \|\frac{\mathrm{d}q}{\mathrm{d}t}\|_{L^2(0,T;V')}^2.$$

Now

$$B \in \mathcal{L}(X, Y')$$

is boundedly invertible.

Choice of spaces and verification of the first assumption

We choose

$$\begin{aligned} H &= L^2(\Omega), \\ V &= H^1(\Omega) \end{aligned}$$

which leads to

$$\begin{aligned} X &= L^2(0,1; H^1(\Omega)), \\ Y &= L^2(0,1; H^1(\Omega)) \cap H^1_{\{T\}}(0,T; (H^1(\Omega))'). \end{aligned}$$

Additionally we choose $X'_+ \hookrightarrow X'$ and $Y_+ \hookrightarrow Y$ to be

$$\begin{aligned} X'_+ &= L^2(0,T; L^2(\Omega)) \\ Y_+ &= L^2(0,T; H^2(\Omega)) \cap H^1_{\{T\}}(0,T; L^2(\Omega)) \end{aligned}$$

If we have Dirichlet boundary conditions then they need to be included into the space V.

Verification of the first assumption

For this choice of spaces and under the assumption that $\partial\Omega$ is smooth and that $A(t)$ has smooth coefficients, the operator B fulfils the first assumption, i.e. the regularity condition

$$(B')^{-1} \in \mathcal{L}(X'_+, Y_+).$$

Discretisation and Verification of the second assumption

Tensor product discretisation

The function spaces on the space-time are discretised with a tensor product structure. We choose sequences of discrete subspaces

$$
\begin{aligned}
S^x_{j_0} \subset S^x_{j_0+1} \subset \ldots \subset H^1(\Omega), \\
S^t_{j_0} \subset S^t_{j_0+1} \subset \ldots \subset L^2(0,T)
\end{aligned}
$$

and

$$
\begin{aligned}
Q^x_{l_0} \subset Q^x_{l_0+1} \subset \ldots \subset H^2(\Omega) \\
Q^t_{l_0} \subset Q^t_{l_0+1} \subset \ldots \subset H^1_{\{T\}}(0,T).
\end{aligned}
$$

and define the discrete space-time spaces as

$$
\begin{aligned}
S_j &= S^r_j \otimes S^x_j \subset L^2(0,T;H^1(\Omega)) = X \\
Q_l &= Q^t_l \otimes Q^x_l \subset L^2(0,T;H^2(\Omega)) \cap H^1_{\{T\}}(0,T;L^2(\Omega)) = Y_+
\end{aligned}
$$

Assumptions on the spatial and time spaces

- In the sequel we are going to see which assumptions have to be made on S_j^x, S_j^t, Q_l^t, Q_l^x for their tensor product spaces to fulfil the second assumption of the main theorem.
- We will for now set $\tilde{S}_j = S_j$ but this can be generalised to other choices of $\{S_j\} \subset X$ and $\{\tilde{S}_j\} \subset X'_+$.
- The Jackson and Bernstein inequalities on the tensor product space follow from the respective inequalities for each direction.

Jackson inequality

First, on each of the individual spaces we need the Jackson inequality: For a fixed number $\omega > 1$

$$\inf_{u_j^x \in S_j^x} \|u^x - u_j^x\|_{L^2(\Omega)} \lesssim \omega^{-sj} \|u^x\|_{H^s(\Omega)} \text{ for } u^x \in H^s(\Omega), \qquad 0 \leq s \leq d_S$$

$$\inf_{u_j^t \in S_j^t} \|u^t - u_j^t\|_{L^2(0,T)} \lesssim \omega^{-sj} \|u^t\|_{H^s(0,T)} \text{ for } u^t \in H^s(0,T), \quad 0 \leq s \leq d_S$$

$$\inf_{q_l^x \in Q_l^x} \|q^x - q_l^x\|_{L^2(\Omega)} \lesssim \omega^{-sl} \|q^x\|_{H^s(\Omega)} \text{ for } q^x \in H^s(\Omega), \qquad 0 \leq s \leq d_Q$$

$$\inf_{q_l^t \in Q_l^t} \|q^t - q_l^t\|_{L^2(0,T)} \lesssim \omega^{-sl} \|q^t\|_{H^s(0,T)} \text{ for } q^t \in H^s(0,T), \quad 0 \leq s \leq d_Q$$

for constants d_{S^x}, d_{S^t}, d_{Q^x} and d_{Q^t}, which remain to be determined.

Bernstein inequality

Second, we need the Bernstein inequality on each space:

$$\begin{aligned}
\|u_j^x\|_{H^s(\Omega)} &\lesssim \omega^{sj}\|u_j^x\|_{L^2(\Omega)} \text{ for } u_j^x \in S_j^x, \quad 0 \le s < \gamma_{S^x},\\
\|u_j^t\|_{H^s(0,T)} &\lesssim \omega^{sj}\|u_j^t\|_{L^2(0,T)} \text{ for } u_j^t \in S_j^t, \quad 0 \le s < \gamma_{S^t},\\
\|q_l^x\|_{H^s(\Omega)} &\lesssim \omega^{sl}\|q_l^x\|_{L^2(\Omega)} \text{ for } q_l^x \in Q_l^x, \quad 0 \le s < \gamma_{Q^x},\\
\|q_l^t\|_{H^s(0,T)} &\lesssim \omega^{sl}\|q_l^t\|_{L^2(0,T)} \text{ for } q_l^t \in Q_l^t, \quad 0 \le s < \gamma_{Q^t}.
\end{aligned}$$

Choice of the constants

The last assumptions to be made are those on the constants d_F, γ_F in the Bernstein and Jackson inequalities. We assume

$$
\begin{aligned}
\gamma_{S^x}, d_{S^x} &> 1, \\
\gamma_{S^t}, d_{S^t} &> 0, \\
\gamma_{Q^x}, d_{Q^x} &> 2 \\
\gamma_{Q^t}, d_{Q^t} &> 1.
\end{aligned}
$$

For example, these assumptions can be fulfilled for hierachical (h-refined) spline spaces

Conclusion

- If each direction of a tensor product discretisation fulfils the Jackson and Bernstein inequalities with the given constants γ_F and d_F, then also the tensor product fulfils the Jackson and Berstein inequalities.
- The assumptions of the main theorem for $\tilde{S}_j = S_j$ are thus fulfilled in this case.
- If we want to choose $S_j \neq \tilde{S}_j \subset X'_+$, we need an additional stability assumption as well as the Jackson and Berstein inequalities for $\tilde{S}^x_j$ and $\tilde{S}^t_j$.