# A Space-Time Petrov-Galerkin method for linear wave equations

## Talk No. 2

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# Introduction: Systems of linear conservation laws

The examples that were discussed in the first talk of this seminar session can be written as systems of linear conservation laws. We can write

$$Av = \sum_{d=1}^{D} B_d \partial_d v = \mathrm{div} F(v)$$

for symmetric $J \times J$-matrices $B_d$ and linear flux function

$$F(v) = [B_1 v, \dots, B_D v].$$

The problem then becomes

$$M\partial_t u(t) + \mathrm{div} F(u(t)) = f(t) \quad \text{for } t \in [0, T]$$
$$u(0) = u_0.$$

# Overview

## Defining the numerical flux

## Discretisation

- Discontinuous Galerkin discretisation in space
- Petrov-Galerkin discretisation in space-time

## Existence, uniqueness and error convergence

# Defining the numerical flux

# Numerical flux

In our numerical method we are going to need to replace the flux

$$n \cdot F(v)$$

between two elements by a numerical flux

$$n \cdot F^{\mathrm{up}}(v).$$

We will define $F^{\mathrm{up}}$ using local solutions of the Riemann problem.

# Weak formulation

A function

$$u \in \mathrm{L}^1((0,T) \times \mathbb{R}^D, \mathbb{R}^J)$$

is a weak solution to our problem with right hand side $f = 0$ if for all $\phi \in \mathrm{C}^1_0((-1,T) \times \Omega, \mathbb{R}^J)$

$$\int_{(0,T)\times\mathbb{R}^D} uM\partial_t\phi \mathrm{d}t\mathrm{d}x + \int_{(0,T)\times\mathbb{R}^D} u\mathrm{div}F(\phi)\mathrm{d}t\mathrm{d}x$$
$$+ \int_{\mathbb{R}^D} Mu_0(x)\phi(0,x)\mathrm{d}x = 0$$

# The Riemann problem

For a unit vector $n \in \mathbb{R}^D$ and the piecewise constant initial function

$$u_0(x) = \begin{cases} u_L & \text{if } n \cdot x < 0 \\ u_R & \text{if } n \cdot x > 0 \end{cases}$$

with $u_L, u_R \in \mathbb{R}^J$ find a piecewise constant weak solution with right hand side $f = 0$.

# Solution to the Riemann problem

Begin with constructing discontinuous traveling waves.
For any unit vector $n = (n_1, \ldots, n_D)$ the flux is given by

$$n \cdot F(u) = Bu$$

with the symmetric matrix

$$B = \sum_{d=1}^{D} n_d B_d.$$

For any eigenpair $(\lambda, w) \in \mathbb{R} \times \mathbb{R}^J$ with

$$Bw = \lambda M w$$

the piecewise constant function $u$ on $[0, T] \times \mathbb{R}^D$ given by

$$u(t,x) = \begin{cases} \alpha_L w & \text{if } n \cdot x - \lambda t < 0 \\ \alpha_R w & \text{if } n \cdot x - \lambda t > 0 \end{cases}$$

with $\alpha_L, \alpha_R \in \mathbb{R}$ is a weak solution.

We superpose traveling waves to get a solution to the Riemann problem.
Let $(\lambda_j, w_j)$ be $M$-orthonormal eigenpairs of $B$, i.e.

$$w_k \cdot M w_j = \delta^{kj}.$$

Then

$$u(t,x) = \sum_{j=1}^{J} a_j(x \cdot n - \lambda_j t) w_j$$

with

$$a_j(s) = \begin{cases} w_j \cdot M u_L & \text{if } s < 0 \\ w_j \cdot M u_R & \text{if } s > 0 \end{cases}$$

is a solution to the Riemann problem.

# Definition of the upwind flux

The solution to the Riemann problem at $(t, 0)$ for $t > 0$ defines the upwind flux on $\partial\Omega_L \cap \partial\Omega_R$, where

$$\Omega_L = \{x \in \mathbb{R}^D : n \cdot x < 0\},$$
$$\Omega_R = \{x \in \mathbb{R}^D : n \cdot x > 0\}.$$

It is

$$n \cdot F^{\mathrm{up}}(u_0) = Bu_L + \sum_{\lambda_j < 0} w_j \cdot M[u]Bw_j,$$

where $[u] = u_R - u_L$ is the jump term.

# Discretisation

# Discontinuous Galerkin discretisation in space

We assume to be $\Omega$ a bounded polyhedral Lipschitz domain that is decomposed into open elements $K \subset \Omega$ such that

$$\bar{\Omega} = \bigcup_{K \in \mathcal{K}} \bar{K}.$$

We denote by

- $\mathcal{K}$ the set of elements,
- $\mathcal{F}_K$ the faces of an element $K$,
- $n_K$ the outer unit normal on $\partial K$.

We choose polynomial degrees $p_K$ on each element and define the global discontinuous Galerkin space

$$H_h = \{v_h \in \mathrm{L}^2(\Omega)^J : \forall K \in \mathcal{K} \quad v_h|_K \in H_{h,K} = \mathbb{P}_{p_K}(K, \mathbb{R}^J)\}$$

# Discretisation of the mass operator

The mass operator $M_h \in \mathcal{L}(H_h, H_h)$ is defined as the Galerkin approximation of $M$, i.e.

$$(M_h v_h, w_h)_{0,\Omega} = (M v_h, w_h)_{0,\Omega} \quad \forall v_h, w_h \in H_h.$$

# Discretisation of the differential operator

We define the discrete operator $A_h \in \mathcal{L}(H_h, H_h)$ on each element $K$ for $v_h \in H_h$ and $\phi_{h,K} \in H_{h,k}$ using the numerical flux:

$$\begin{aligned}(A_h v_h, \phi_{h,K})_{0,K} =& (\mathrm{div} F(v_{h,K}), \phi_{h,K})_{0,K} \\ &+ \sum_{f \in \mathcal{F}_K} (n_K \cdot (F_K^{\mathrm{up}}(v_h) - F(v_{h,K})), \phi_{h,K})_{0,f}.\end{aligned}$$

- This definition, in our applications, is consistent, i.e. for all $v \in D(A)$ and $\phi_h \in H_h$

$$(Av, \phi_h)_{0,\Omega} = (A_h v, \phi_h)_{0,\Omega}$$

and for all $v_h \in H_h$, $v \in D(A) \cap \mathrm{H}^1(\Omega, \mathbb{R}^J)$

$$\sum_{K \in \mathcal{K}} (n_K \cdot F_K^{\mathrm{up}}(v_{h,K}), v)_{0,\partial K} = 0.$$

- For all our applications there is a constant $C > 0$ such that

$$(A_h v_h, v_h)_{0,\Omega} \geq C \sum_{K \in \mathcal{K}} \sum_{f \in \mathcal{F}_K} ||n_K(\cdot(F_K^{\mathrm{up}}(v_h) - F(v_{h_K}))||_{0,f}^2 \geq 0.$$

# Petrov-Galerkin discretisation in space-time

Next, we decompose the space-time cylinder $Q = [0, T] \times \Omega$ using a tensor-product discretisation. We choose a time sequence

$$0 = t_0 < \ldots < t_N = T$$

and define elements $R = (t_{n-1}, t_n) \times K$ for $K \in \mathcal{K}$. Then

$$\bar{Q} = \bigcup_{R \in \mathcal{R}} \bar{R}.$$

Now we want to define ansatz and test spaces $V_h$ and $W_h$ such that on each element $R \in \mathcal{R}$ the local spaces fulfil

$$W_{h,R} \subset \partial_t V_{h,R}.$$

On each element $R = (t_{n-1}, t_n) \times K$ we choose as the local test space

$$W_{h,R} = H_{h,K}$$

to be constant in time. The global test space then is

$$W_h = \{w_h \in \mathrm{L}^2((0, T), H) : \forall n \; w_h(t_n, x) \in H_h \text{ and } \\ w_h(t, x) = w_h(t_{n-1}, x) \text{ on } (t_{n-1}, t_n)\}.$$

For this choice of test spaces the global ansatz space then consists of functions which are piecewise linear in time:

$$V_h = \{v_h \in \mathrm{H}^1((0,T),H) : v_h(0,x) = 0, \forall n \; v_h(t_n,x) \in H_h \text{ and}$$
$$v_h(t,x) = \frac{t_n - t}{t_n - t_{n-1}} v_h(t_{n-1},x) + \frac{t - t_{n-1}}{t_n - t_{n-1}} v_h(t_n,x) \text{ on } (t_{n-1},t_n)\}$$

Functions in $V_h$ are continuous in time.

Remark

*We can generalise this for any polynomial degree in time.*

# The discrete space-time operator

We extend the differential operator $A_h$ to a space-time operator in $\mathcal{L}(V_h, W_h)$ by defining on each element $R = (t_{n-1}, t_n) \times K$

$$
\begin{aligned}
(A_h v_h, w_h)_{0,R} =& (\mathrm{div} F(v_{h,R}), w_{h,R})_{0,R} \\
&+ \sum_{f \in \mathcal{F}_K} n_K \cdot (F_K^{\mathrm{up}}(v_h) - F(v_{h,R})), w_{h,R})_{0,(t_{n-1},t_n)\times f}\,.
\end{aligned}
$$

Finally we can define the discrete space-time operator $L_h \in \mathcal{L}(V_h, W_h)$ and the corresponding bilinear form $b_h$ by

$$b_h(v_h, w_h) = (L_h v_h, w_h)_{0,Q} = (M_h \partial_t v_h + A_h v_h, w_h)_{0,Q}.$$

# Existence, uniqueness and error convergence

# Existence and uniqueness

As in the infinite-dimensional case we use the norm

$$||w_h||_W^2 = (Mw_h, w_h)_{0,Q}$$

on $W_h$ and the norm

$$||v_h||_{V_h}^2 = ||v_h||_W^2 + ||M_h^{-1} L_h v_h||_W^2$$

on $V_h$.
Again we are going to prove the inf-sup stability of $b_h$ with respect to these norms.

First, we need the following lemma

### Lemma (No proof)

*For any $v_h \in V_h$ we have*

$$(M_h \partial t v_h, d_T v_h)_{0,Q} \leq (L_h v_h, d_T \Pi_h v_h)_{0,Q},$$

*where $\Pi_h : W \to W_h$ is the $L_2$-projection and $d_T(t) = (T - t)$.*

The proof involves the non-negativity of $A_h$ and the tensor-product structure of the discretisation.

### Lemma (inf-sup stability)

*$b_h$ is continuous and fulfils for all $v_h \in V_h$*

$$\sup_{w_h \in W_h \setminus \{0\}} \frac{b_h(v_h, w_h)}{||w_h||_W} \geq \beta ||v_h||_{V_h}$$

*with $\beta = (1 + 4T^2)^{-\frac{1}{2}}$.*

### Theorem (Existence and uniqueness)

*For any* $f \in \mathrm{L}^2(Q, \mathbb{R}^J)$ *there exists a unique solution* $u_h \in V_h$ *such that*

$$b_h(u_h, w_h)_{0,Q} = (f, w_h)_{0,Q}$$

*for all* $w_h \in W_h$.

# Convergence

### Theorem

*Let $u \in V$ be the solution of the original problem and $u_h \in V_h$ the solution of the discrete problem. Then*

$$||u - u_h||_{V_h} \leq (1 + \beta^{-1}) \inf_{v_h \in V_h} ||u - v_h||_{V_h}.$$

*If the solution is sufficiently smooth, then*

$$||u - u_h||_{V_h} \leq C(\Delta t + \Delta x^p)(||\partial_t u||_{0,Q} + ||D^{p+1} u||_{0,Q}),$$

*where $\Delta t, \Delta x, p \geq 1$ with $p \leq p_K$, $\Delta t \geq (t_{n-1} - t_n)$ and $\Delta x \geq \mathrm{diam}(K)$.*