First Initial-Boundary Value Problem for General Parabolic Equations

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The Problem

Let $\mathcal{Q}_T = \Omega \times (0, T)$ be a space-time cylinder with a sufficiently smooth boundary. We consider the problem

$$\begin{aligned} \mathcal{M}u \equiv & u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i(x,t)u) \\ & + b_i(x,t)u_{x_i} + a(x,t)u = f + \frac{\partial f_i}{\partial x_i} \end{aligned} \tag{1}$$

$$u|_{t=0} = \varphi(x), \qquad u|_{S_T} = 0 \tag{2}$$

with the conditions $a_{ij} = a_{ji}$,

$$\sqrt{\sum_{i=1}^{n} a_i^2}, \quad \sqrt{\sum_{i=1}^{n} b_i^2}, \quad |a| \le \mu, \tag{3}$$

$$\varphi \in L_2(\Omega), \quad f \in L_{2,1}(\mathcal{Q}_T), \quad f_i \in L_2(\mathcal{Q}_T) \tag{4}$$

and

$$\nu\xi^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mu\xi^2, \quad \nu, \mu = const > 0. \tag{5}$$

Motivation

Consider an electrical machine with the follwing assumptions:

1. $\hat{\Omega} = \Omega \times (-\infty, +\infty)$
2. $J_i = (0, 0, J_3(x_1, x_2))^T$
3. $M = H_0 = (H_{01}(x_1, x_2), H_{02}(x_1, x_2), 0)^T$,
   $H = (H_1(x_1, x_2), H_2(x_1, x_2), 0)^T$

With the ansatz $B = \mathrm{curl} A$, where $A = (0, 0, u(x_1, x_2))^T$, we obtain the equations

$$u_t - \frac{\partial}{\partial x_i}(\nu(x,t) u_{x_i}) = J_3(x,t) - (\frac{\partial H_{01}}{x_2} - \frac{\partial H_{02}}{x_1})$$

Roadmap

1. Derive an a priori bound for $|u|_{\mathcal{Q}_t}$
2. Proof of solvability in the space $H^{1,0}(\mathcal{Q}_T)$
3. Show that such a generalized solution is in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$
4. Proof of uniqueness in $H^{1,0}(\mathcal{Q}_T)$

Deriving of a bound for $|u|_{\mathcal{Q}_t}$

To get a bound for the norm $|u|_{\mathcal{Q}_t}$, we need to derive an energy balance equation. We follow the procedure:

- Multiply the PDE with u

$$(\mathcal{M}u)u = (f + \frac{\partial f_i}{\partial x_i})u$$

- Integrate over the domain $\mathcal{Q}_t := \Omega \times (0, t)$

$$\int_{\mathcal{Q}_t} (\mathcal{M}u)u \, \mathrm{d}x\mathrm{d}t = \int_{\mathcal{Q}_t} (f + \frac{\partial f_i}{\partial x_i})u \, \mathrm{d}x\mathrm{d}t$$

An Energy Balance Equation

We continue with

- Integration by parts and incorporation of the boundary conditions
- Rewriting the time derivative and obtaining the *energy balance equation*

$$\frac{1}{2}\|u(\cdot,t)\|_{2,\Omega}^2 + \int_{\mathcal{Q}_t} (a_{ij}u_{x_j}u_{x_i} + a_i u u_{x_i} + b_i u_{x_i} u + a u^2)\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\|u(\cdot,0)\|_{2,\Omega}^2 + \int_{\mathcal{Q}_t} (fu - f_i u_{x_i})\,\mathrm{d}x\mathrm{d}t \quad (6)$$

We will use this equation to derive a bound for

$$|u|_{\mathcal{Q}_t} := \max_{0\le\tau\le t} \|u(\cdot,\tau)\|_{2,\Omega} + \|u_x\|_{2,\mathcal{Q}_t}.$$

Deriving the bound

From (6) we obtain

$$\begin{aligned}
&\frac{1}{2}\|u(\cdot,t)\|^2_{2,\Omega} + \nu\|u_x\|^2_{2,\mathcal{Q}_t} \\
&\qquad \leq \frac{1}{2}\|u(\cdot,0)\|^2_{2,\Omega} + \frac{\nu}{2}\|u_x\|^2_{2,\mathcal{Q}_t} + (\frac{2\mu^2}{\nu} + \mu)\|u\|^2_{2,\mathcal{Q}_t} \\
&\qquad\quad + \|f\|_{2,1,\mathcal{Q}_t} \max_{0\leq\tau\leq t} \|u(\cdot,\tau)\|_{2,\Omega} + \|\boldsymbol{f}\|_{2,\mathcal{Q}_t}\|u_x\|_{2,\mathcal{Q}_t}.
\end{aligned}$$

With some rewriting, we obtain then

$$\begin{aligned}
&\|u(\cdot,t)\|^2_{2,\Omega} + \|u_x\|^2_{2,\mathcal{Q}_t} \\
&\qquad \leq y(t)\|u(\cdot,0)\|_{2,\Omega} + cty(t)^2 \\
&\qquad\quad + 2y(t)\|f\|_{2,1,\mathcal{Q}_t} + 2\|\boldsymbol{f}\|_{2,\mathcal{Q}_t}\|u_x\|_{2,\mathcal{Q}_t} \equiv j(t)
\end{aligned}$$

with $y(t) = \max_{0\leq\tau\leq t} \|u(\cdot,\tau)\|_{2,\Omega}$ and $c = 2((2\mu^2)/\nu + \mu)$.

With this equation, we further obtain two inequalities

$$
\begin{aligned}
y(t)^2 &\leq j(t), \\
\|u_x\|_{2,\mathcal{Q}_t}^2 &\leq \nu^{-1} j(t).
\end{aligned}
$$

Taking the square root and adding them up yields

$$
\begin{aligned}
|u|_{\mathcal{Q}_t} \equiv y(t) + \|u_x\|_{2,\mathcal{Q}_t} \\
\leq & (1+\nu^{-\frac{1}{2}})\sqrt{ct}|u|_{\mathcal{Q}_t} + (1+\nu^{-\frac{1}{2}})|u|_{\mathcal{Q}_t}^{-1/2} \\
& \times \left(\|u(\cdot,0)\|_{2,\Omega} + 2\|f\|_{2,1,\mathcal{Q}_t} + \|\boldsymbol{f}\|_{2,\mathcal{Q}_t}\right)^{1/2}
\end{aligned}
$$

For $t < t_1 \equiv (1+\nu^{-\frac{1}{2}})^{-2}c^{-1}$, we get the estimate

$$
\begin{aligned}
|u|_{\mathcal{Q}_t} \leq & (1+\nu^{-\frac{1}{2}})^2(1-(1+\nu^{-\frac{1}{2}})\sqrt{ct})^{-2} \\
& \times \left(\|u(\cdot,0)\|_{2,\Omega} + 2\|f\|_{2,1,\mathcal{Q}_t} + \|\boldsymbol{f}\|_{2,\mathcal{Q}_t}\right)
\end{aligned}
$$

We subdivide out time interval $[0, t]$ into subintervals Δ_l, i.e, $\Delta_1 = [0, \frac{1}{2}t_1]$, $\Delta_2 = [\frac{1}{2}t_1, t_1]$, ..., where length of the last interval Δ_N does not exceed $t_1/2$. On each of these subintervals we have now such a bound, therefore we obtain

$$|u|_{\mathcal{Q}_t} \le c_1(t)\mathcal{F}(t), \quad \forall t \in [0, T] \tag{7}$$

with the constant $c_1(t)$ depending on μ, ν and t and with

$$\mathcal{F}(t) = \|u(\cdot, 0)\|_{2,\Omega} + 2\|f\|_{2,1,\mathcal{Q}_t} + \|\boldsymbol{f}\|_{2,\mathcal{Q}_T}.$$

Definition of a Generalized Solution

Definition

We call $u \in H^{1,0}(\mathcal{Q}_T)$ a *generalized solution in* $H^{1,0}(\mathcal{Q}_T)$ (or $\mathring{H}^{1,0}(\mathcal{Q}_T)$), if it satisfies the integral identity

$$\begin{aligned}\mathcal{M}(u,\eta) \equiv \int_{\mathcal{Q}_T} \left(-u\eta_t + a_{ij}u_{x_j}\eta_{x_i} + a_i u\eta_{x_i} + b_i u_{x_i}\eta + au\eta\right) \mathrm{d}x\mathrm{d}t \\ = \int_\Omega \varphi\eta(x,0)\,\mathrm{d}x + \int_{\mathcal{Q}_T} \left(f\eta - f_i\eta_{x_i}\right) \mathrm{d}x\mathrm{d}t \qquad (8)\end{aligned}$$

for all $\eta \in \hat{H}^1_0(\mathcal{Q}_T) := \{v \in H^1_0(\mathcal{Q}_T) : v|_{t=T} = 0\}$.

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Does our problem have a generalized solution in $\mathring{H}^{1,0}(\mathcal{Q}_T)$?

Proof of Solvability

We use a Galerkin approach. We take a fundamental system $\{\varphi_k(x)\}$ in $\mathring{H}^1(\Omega)$, which has been orthonormalized w.r. to $L_2(\Omega)$. Moreover, let $u^N(x,t) = \sum_{k=1}^N c_k^N(t)\varphi_k(x)$ be an approximate solution of the system

$$\begin{aligned}(u_t^N, \varphi_l) + (a_{ij}u_{x_i}^N + a_i u^N, (\varphi_l)_{x_i}) + (b_i u_{x_i}^N + a u^N, \varphi_l)& \\ = (f, \varphi_l) - (f_i, (\varphi_l)_{x_i}), \quad l = 1, \ldots, N,& \\ c_l^N(0) = (\varphi, \varphi_l)&\end{aligned} \tag{9}$$

This is a system of N ODEs for $c_l(t) \equiv c_l^N(t)$, with principal terms $dc_l(t)/dt$. This system has a unique solution $c_l^N(t)$ on $[0, T]$.

A bound for u^N

Next, we multiply each equation in (9) with the corresponding c_l^N, sum them all up and integrate w.r.t. t from 0 to $t \leq T$:

$$\int_{\mathcal{Q}_t} (u_t^N u^N + a_{ij} u_{x_j}^N u_{x_i}^N + a_i u^N u_{x_i}^N + b_i u_{x_i}^N u^N + a(u^N)^2) \,\mathrm{d}x\mathrm{d}t$$

$$= \int_{\mathcal{Q}_t} (f u^N - f_i u_{x_i}^N) \,\mathrm{d}x\mathrm{d}t$$

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As before, the energy balance equation holds and moreover the bound

$$|u^N|_{\mathcal{Q}_t} \leq c_1(t),$$

where $c_1(t)$ is a constant independent of N.

The sequence $\{u^N\}$ is bounded, it has a subsequence $\{u^{N_k}\}$, that converges weakly, together with the derivatives $u^{N_k}_{x_i}$, in $L_2(Q_T)$ to some element $u(x,t) \in \mathring{H}^{1,0}(Q_T)$. This element $u(x,t)$ is the desired generalized solution.

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We multiply each equation in (9) with some arbitrary, absolutely continuous function $d_l(t)$, with $dd_l/dt \in L_2(0,T)$ and $d_l(T) = 0$. Again, we add them up, integrate over $(0,T)$ and obtain the result

$$\mathcal{M}(u^N, \mathbf{\Phi}) = \int_\Omega \varphi \mathbf{\Phi}|_{t=0} \,\mathrm{d}x + \int_{\mathcal{Q}_T} (f\mathbf{\Phi} - f_i \mathbf{\Phi}_{x_i}) \,\mathrm{d}x\mathrm{d}t$$

with $\mathcal{M}(\cdot,\cdot)$ as before and $\mathbf{\Phi}(x,t) = \sum_{l=1}^N d_l(t)\varphi_l(x)$.

We denote by $\mathfrak{M}_N$ the set of functions $d_l(t)$, $l = 1, ..., N$, which fulfil the conditions above. The totality $\bigcup_{p=1}^{\infty} \mathfrak{M}_p$ is dense in the subspace $\hat{H}_0^1(\mathcal{Q}_T)$ of $H_0^1(\mathcal{Q}_T)$.

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We fix now a $\mathbf{\Phi} \in \mathfrak{M}_p$ and take the limit of the subsequence $\{u^{N_k}\}$, starting with $N_k \geq p$. We obtain the definition of a generalized solution (8) for $u(x,t)$, with $\eta = \mathbf{\Phi} \in \mathfrak{M}_p$. As the union $\bigcup_{p=1}^{\infty} \mathfrak{M}_p$ is dense in $\hat{H}_0^1(\mathcal{Q}_T)$, the definition holds for any $\eta \in \hat{H}_0^1(\mathcal{Q}_T)$, that means $u(x,t)$ is indeed a generalized solution in $\overset{\circ}{H}{}^{1,0}(\mathcal{Q}_T)$.

An Existence Theorem

Theorem 3.1

If the conditions $a_{ij} = a_{ji}$,

$$\sqrt{\sum_{i=1}^{n} a_i^2}, \quad \sqrt{\sum_{i=1}^{n} b_i^2}, \quad |a| \leq \mu,$$

$$\varphi \in L_2(\Omega), \quad f \in L_{2,1}(Q_T), \quad f_i \in L_2(Q_T)$$

and

$$\nu\xi^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \mu\xi^2, \quad \nu, \mu = const > 0.$$

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But is this solution unique?

Uniqueness of a Solution

We will now use the following two theorems from the last presentation.

Theorem 2.2

The problem

$$u_t - \Delta u = f + \frac{\partial f_i}{\partial x_i}$$
$$u|_{t=0} = \varphi(x), \qquad u|_{S_T} = 0$$

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Theorem 2.3

The problem in Thm. 2.2 has a generalized solution in $\overset{\circ}{V}{}_2^{1,0}(Q_T)$ for $\varphi \in L_2(\Omega)$, $f \in L_{2,1}(Q_T)$, and $f_i \in L_2(Q_T)$.

Now we consider our generalized solution $u \in \mathring{H}^{1,0}(\mathcal{Q}_T)$ as a generalized solution in $L_2(\mathcal{Q}_T)$ for the problem

$$u_t - \Delta u = \tilde{f} + \frac{\partial \tilde{f}_i}{\partial x_i} \tag{10}$$

$$u|_{t=0} = \varphi(x), \qquad u|_{S_T} = 0 \tag{11}$$

with $\tilde{f} = f - b_i u_{x_i} - au$ and $\tilde{f}_i = f_i + a_{ij} u_{x_j} + a_i u - u_{x_i}$. It holds that $\tilde{f} \in L_{2,1}(\mathcal{Q}_T)$ and $\tilde{f}_i \in L_2(\mathcal{Q}_T)$.

We transform the identity

$$\mathcal{M}(u,\eta) = \int_\Omega \varphi\eta(x,0)\,\mathrm{d}x + \int_{\mathcal{Q}_T} (f\eta - f_i\eta_{x_i})\,\mathrm{d}x\mathrm{d}t$$

to

$$\int_{\mathcal{Q}_T} u(\eta_t + \Delta\eta)\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi\eta(x,0)\,\mathrm{d}x = \int_{\mathcal{Q}_T} (-\tilde{f}\eta + \tilde{f}_i\eta_{x_i})\,\mathrm{d}x\mathrm{d}t \quad (12)$$

which holds for all $H_0^{\Delta,1}(\mathcal{Q}_T)$ with $\eta|_{t=T} = 0$.

Then it follows from Thm. 2.2 that u is indeed unique in $L_2(\mathcal{Q}_T)$ and from Thm. 2.3 the problem (12) has a generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$.

Then it follows from Thm. 2.2 that u is indeed unique in $L_2(\mathcal{Q}_T)$ and from Thm. 2.3 the problem (12) has a generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$. Moreover, it holds that

$$\frac{1}{2}\|u(\cdot,t)\|_{2,\Omega}^2 + \|u_x\|_{2,\mathcal{Q}_t}^2 = \frac{1}{2}\|u(\cdot,0)\|_{2,\Omega}^2 + \int_{\mathcal{Q}_t} (\tilde{f}u - \tilde{f}_i u_{x_i})\,\mathrm{d}x\mathrm{d}t \tag{13}$$

and

$$\int_\Omega u(x,t)\eta(x,t)\,\mathrm{d}x - \int_\Omega \varphi\eta(x,0)\,\mathrm{d}x \\ + \int_{\mathcal{Q}_t}(-u\eta_t + u_x\eta_x)\,\mathrm{d}x\mathrm{d}t = \int_{\mathcal{Q}_t}(\tilde{f}\eta - \tilde{f}_i\eta_{x_i})\,\mathrm{d}x\mathrm{d}t. \tag{14}$$

We can easily rewrite (13) to our energy balance equation (6) and we rewrite (14) to

$$\begin{aligned}\int_\Omega u(x,t)\eta(x,t)\,\mathrm{d}x - \int_\Omega \varphi\eta(x,0)\,\mathrm{d}x \\ + \int_{\mathcal{Q}_t} (-u\eta_t + a_{ij}u_{x_j}\eta_{x_i} + a_i u\eta_{x_i} + b_i u_{x_i}\eta + au\eta)\,\mathrm{d}x\mathrm{d}t \\ = \int_{\mathcal{Q}_t} (f\eta - f_i\eta_{x_i})\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{15}$$

We have now proved that any generalized solution in $\overset{\circ}{H}{}^{1,0}(\mathcal{Q}_T)$ is a generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$ and they fulfil the energy balance equation (6) and (15). Are these solutions unique?

Suppose we have two generalized solutions $u' \neq u''$ for the same right hand side and initial data, then the difference $u = u' - u''$ is also a generalized solution in $\overset{\circ}{H}{}^{1,0}(\mathcal{Q}_T)$ with zero right hand side and homogeneous initial data. We have shown that it is then also a generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$ and fulfils

$$\frac{1}{2}\|u(\cdot,t)\|_{2,\Omega}^2 + \|u_x\|_{2,\mathcal{Q}_t}^2 = 0$$

as well as

$$|u|_{\mathcal{Q}_t} \leq 0.$$

From this, we deduce that $u = 0$ and moreover $u' = u''$, hence it is unique in $H^{1,0}(\mathcal{Q}_T)$.

An Uniqueness Theorem

With these arguments, it follows that the operator B, which assigns each $\{f; f_i; \varphi\}$ its generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$ is linear and that the energy balance equation is a result of (15).

Theorem

Under the assumptions of Theorem 3.1, any generalized solution from $H^{1,0}(\mathcal{Q}_T)$ is the generalized solution in $\overset{\circ}{V}{}_2^{1,0}(\mathcal{Q}_T)$ and it is unique in $H^{1,0}(\mathcal{Q}_T)$.

Thank you for your attention!