Overview of parabolic space-time methods

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Seminar: Space-Time Methods for PDEs

Outline

- Katharina: Continuous variational formulations
  - Ladyzhenskaya
  - Mollet

  - Schwab and Stevenson, Andreev
  - Steinbach

- Christoph: Discretization methods

Parabolic problem

Find u such that

$$\frac{\partial u}{\partial t}(x,t) + Lu(x,t) = f(x,t) \quad \text{for } (x,t) \in Q_T = \Omega \times (0,T),$$

with

$$u|_{t=0} = u_0(x) \quad \text{for } x \in \Omega,$$
$$u = 0 \quad \text{for } (x,t) \in \Sigma_T = \partial\Omega \times [0,T].$$

Example:

$$Lu = -\operatorname{div}_x(A(x,t)\nabla_x u(x,t)).$$

Ladyzhenskaya

Find $u \in X$ s.t.

$$b(u,v) = f(v) \quad \text{for all } v \in Y$$

with

$$\begin{aligned} X &= L_2(0,T;H_0^1(\Omega)) = \overset{\circ}{W}{}_2^{1,0}(Q_T) \\ Y &= \hat{H}_0^1(Q_T) \\ b(u,v) &= \int_{Q_T} -u(x,t)\frac{\partial v}{\partial t}(x,t) + A(x,t)\nabla_x u(x,t) \cdot \nabla_x v(x,t)\; dx\; dt \\ f(v) &= \int_{Q_T} fv\; dx\; dt + \int_\Omega u_0(x)v(x,0)\; dx \end{aligned}$$

- $H_0^1(Q_T) = \{v \in H^1(Q_T) : v|_{\Sigma_T} = 0\}$
- $\hat{H}_0^1(Q_T) = \{v \in H_0^1(Q_T) : v|_{t=T} = 0\}$
- $f \in L^2(Q_T), u_0 \in L^2(\Omega)$

- No distinction between space and time variables: $v(x,t)$
- For $v \in \hat{H}_0^1(Q_T)$
  - $\frac{\partial v}{\partial t} \in L^2(Q_T)$
  - $v(x,0) \in H^{1/2}(\Omega) \subset L^2(\Omega)$ (standard trace theorem)
- integration by parts w.r.t. t
  $\rightsquigarrow$ initial condition is incorporated as natural condition
- Existence of solution
  - semi-discretization w.r.t. $x \rightsquigarrow$ system of ODE's of 1^{st} order in t
  - boundedness of the sequence
  - reflexive $\rightsquigarrow$ weakly convergent subsequence
  - show limit is solution

In the following $v(x,t) = v(t)(x)$, i.e., $v : (0,T) \to V = H_0^1(\Omega)$

Definition (Bochner space)

$$L^2(0,T;V) = \{v : (0,T) \to V : \text{Bochner-measurable and } \|v\|_{L^2(0,T,V)} < \infty\},$$

where

$$\|v\|^2_{L^2(0,T;V)} = \int_0^T \|v(t)\|_V^2 \, dt.$$

Definition (Bochner-measurable)

A function $v : [0,T] \to V$ is Bochner-measurable iff

$$\forall l \in V^* : t \mapsto \langle l, v(t) \rangle_{V^* \times V} \text{ is Lebesgue-measurable on } (0,T).$$

Analog $L^2(0,T;V^*)$.

Definition

$$H^1(0,T;V^*) = \{v \in L^2(0,T;V^*) : \text{weak derivative } \frac{dv}{dt} \in L^2(0,T;V^*)\}$$

$$L^2(0,T;V) \cap H^1(0,T;V^*) = \{v : v \in L_2(0,T;V), v, \frac{\partial v}{\partial t} \in L_2(0,T;V^*)\}$$

Lemma (trace)

$$L^2(0,T;V) \cap H^1(0,T;V^*) \hookrightarrow C([0,T];H)$$

For $v \in L^2(0,T;V) \cap H^1(0,T;V^*)$, $v(t) \in H$ is well-defined.

Mollet

Find $u \in X$ s.t.

$$b(u,v) = f(v) \quad \text{for all } v \in Y,$$

with

$$\begin{aligned}
X &= L_2(0,T;V) \\
Y &= L_2(0,T;V) \cap H^1_{\{T\}}(0,T;V^*) \\
b(u,v) &= \int_0^T -\langle u(t), \frac{dv}{dt}(t)\rangle_{V\times V^*} + \langle A(t)u(t), v(t)\rangle_{V^*\times V} \, dt \\
f(v) &= \int_0^T \langle f(t), v(t)\rangle_{V^*\times V} \, dt + (u_0, v(0))_H,
\end{aligned}$$

- $H^1_{\{T\}}(0,T;V^*) = H^1(0,T;V^*)$ with $v(T) = 0$

- $v : (0, T) \to V = H^1_0(\Omega)$
- For $v \in L_2(0, T; V) \cap H^1_{\{T\}}(0, T; V^*)$
  - $\frac{dv}{dt} \in L^2(0, T; V^*)$
  - $v(0) \in H = L^2(\Omega)$ well defined (trace lemma)
- integration by parts w.r.t. t
  $\rightsquigarrow$ initial condition is incorporated as natural condition
- $\hat{H}^1_0(Q_T) \subset L_2(0, T; V) \cap H^1_{\{T\}}(0, T; V^*)$
  For $v \in H^1(Q_T)$

$$\|v\|^2_Y \simeq \|v\|^2_{L_2(0,T;V)} + \left\| \frac{dv}{dt} \right\|^2_{L_2(0,T;V^*)}$$
$$= \int_0^T \|v(t)\|^2_{H^1(\Omega)} \, dt + \int_0^T \left\| \frac{dv}{dt}(t) \right\|^2_{H^{-1}(\Omega)} \, dt \leq \|v\|^2_{H^1(Q_T)}$$

Schwab and Stevenson, Andreev

Find $u \in X$ s.t.

$$b(u,v) = f(v) \quad \text{for all } v = (v_1, v_2) \in Y,$$

with

$$\begin{aligned} X &= L_2(0,T;V) \cap H^1(0,T;V^*) \\ Y &= L_2(0,T;V) \times H \\ b(u,(v_1,v_2)) &= \int_0^T \langle \frac{du}{dt}(t), v_1(t) \rangle + \langle A(t)u(t), v_1(t) \rangle \; dt + (u(0), v_2)_H \\ f(v_1,v_2) &= \int_0^T \langle f(t), v_1(t) \rangle \; dt + (u_0, v_2)_H, \end{aligned}$$

$\langle .,. \rangle = \langle .,. \rangle_{V^* \times V}$

- X and Y interchanged (up to initial condition)
- For $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$
  - $\frac{du}{dt} \in L^2(0,T;V^*)$
  - $u(0) \in H = L^2(\Omega)$ well defined (trace lemma)
- no integration by parts w.r.t. t
  $\rightsquigarrow$ initial condition is incorporated in the variational sense
- Existence and uniqueness of solution
  - boundedness
  - inf-sup condition

$$\inf_{u \in X \setminus \{0\}} \sup_{v \in Y \setminus \{0\}} \frac{|b(u,v)|}{\|u\|_X \|v\|_Y} > 0$$

    proof: particular choice $v_1 = u + (A(t)')^{-1} \frac{du}{dt}$ and $v_2 = u(0)$
  - surjectivity

$$\forall v \in Y \setminus \{0\} : \sup_{u \in X \setminus \{0\}} |b(u,v)| > 0$$

    proof: $\rightsquigarrow$ Ladyzhenskaya

Steinbach

Find $u \in X_{u_0}$ s.t.

$$b(u,v) = f(v) \quad \text{for all } v \in Y,$$

with

$$\begin{aligned} X_{u_0} &= \{u \in L_2(0,T;V) \cap H^1(0,T;V^*) : u(0) = u_0 \in H\} \\ Y &= L_2(0,T;V) \\ b(u,v) &= \int_0^T \langle \frac{du}{dt}(t), v(t)\rangle_{V^* \times V} + \langle A(t)u(t), v(t)\rangle_{V^* \times V} \, dt \\ f(v) &= \int_0^T \langle f(t), v(t)\rangle_{V^* \times V} \, dt \end{aligned}$$

- initial condition is incorporated as essential condition into the space

Homogenization

Find $\overline{u} \in X$ s.t.

$$b(\overline{u}, v) = f(v) - b(\overline{u}_0, v) \quad \text{for all } v \in Y,$$

with

$$\begin{aligned} X &= \{\overline{u} \in L_2(0,T;V) \cap H^1(0,T;V^*) : \overline{u}(0) = 0 \in H\} \\ Y &= L_2(0,T;V), \end{aligned}$$

where $\overline{u}_0 \in L_2(0,T;V) \cap H^1(0,T;V^*)$ is some extension of the initial data $u_0 \in H_0^1(\Omega)$.

- $X \subset Y$
- extension of initial data exists due to Schwab and Stevenson (even for $u_0 \in L^2(\Omega)$)

- $X \subset Y$
- Existence and uniqueness of solution
  - boundedness
  - inf-sup condition

$$\inf_{u \in X \setminus \{0\}} \sup_{v \in Y \setminus \{0\}} \frac{|b(u,v)|}{\|u\|_X \|v\|_Y} > 0$$

    proof: particular choice $v = u + N \frac{du}{dt}$
  - surjectivity: not shown !

$$\forall v \in Y \setminus \{0\} : \sup_{u \in X \setminus \{0\}} |b(u,v)| > 0$$