The First Initial-Boundary Value Problem for Hyperbolic Equations

Katharina Rafetseder

Institute of Computational Mathematics
Johannes Kepler University Linz

Seminar: Space-Time Methods for PDEs

Problem

Find u such that

$$u_{tt}(x,t) - Lu(x,t) = f(x,t) \quad \text{for } (x,t) \in Q_T = \Omega \times (0,T),$$

where

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j}) + \sum_{i=1}^{n+1} a_i(x,t)u_{x_i} + a(x,t)u,$$

with

$$u|_{t=0} = \varphi(x) \quad \text{and} \quad u_t|_{t=0} = \psi(x) \quad \text{for } x \in \Omega,$$
$$u = 0 \quad \text{for } (x,t) \in S_T = \partial\Omega \times [0,T].$$

- $f \in L_{2,1}(Q_T)$, $\varphi \in H_0^1(\Omega)$, $\psi \in L_2(\Omega)$
- $$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mu|\xi|^2, \quad \nu > 0$$
  $$\left|\frac{\partial a_{ij}}{\partial t}, a_i, a\right| \le \mu_1$$

Outline

- For solutions $u \in H^2(Q_T)$ show energy inequality

$$z(t)^{\frac{1}{2}} = \left(\int_{\Omega_t} u^2 + u_t^2 + a_{ij} u_{x_i} u_{x_j} dx \right)^{\frac{1}{2}} \leq c_2(t) z^{\frac{1}{2}}(0) + c_3(t) \|f\|_{2,1,Q_t}$$

  (a priori bound for the energy norm in terms of initial data, f)

  $\Rightarrow$ Uniqueness theorem for solutions in $H^2(Q_T)$

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- Uniqueness theorem for generalized solutions in $H^1(Q_T)$

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  (a priori bound for the energy norm in terms of initial data, f)

  $\Rightarrow$ Uniqueness theorem for solutions in $H^2(Q_T)$

- Uniqueness theorem for generalized solutions in $H^1(Q_T)$

- Existence theorem for generalized solution in $H^1(Q_T)$

Energy inequality

$$2\int_{Q_t} (u_{tt} + Lu)u_t \; dx \; dt = 2\int_{Q_t} f u_t \; dx \; dt$$

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Integration by parts for lhs

$$2\int_{Q_t} u_{tt}u_t - \frac{\partial}{\partial x_i}(a_{ij}u_{x_j})u_t + a_i u_{x_i}u_t + a u u_t$$
$$= \int_{Q_t} -\frac{\partial a_{ij}}{\partial t}u_{x_i}u_{x_j} + 2a_i u_{x_i}u_t + 2auu_t\;dx\;dt + \int_{\Omega_t}(u_t^2 + a_{ij}u_{x_i}u_{x_j})dx\Big|_{t=0}^{t=t}$$

Energy inequality

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Integration by parts for lhs

$$2\int_{Q_t} u_{tt}u_t - \frac{\partial}{\partial x_i}(a_{ij}u_{x_j})u_t + a_i u_{x_i}u_t + auu_t$$
$$= \int_{Q_t} -\frac{\partial a_{ij}}{\partial t}u_{x_i}u_{x_j} + 2a_i u_{x_i}u_t + 2auu_t\;dx\;dt + \int_{\Omega_t}(u_t^2 + a_{ij}u_{x_i}u_{x_j})dx\Big|_{t=0}^{t=t}$$

We obtain

$$y(t) = y(0) + \int_{Q_t} \frac{\partial a_{ij}}{\partial t}u_{x_i}u_{x_j} - 2a_i u_{x_i}u_t - 2auu_t + 2fu_t\;dx\;dt,$$

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$$\leq y(0) + c \int_0^t y(t)dt + c_1 \int_{Q_t} u^2 dxdt + 2 \int_0^t \|f\|_{L_2(\Omega_t)} y^{\frac{1}{2}}(t)dt$$

with constants depending on ν and μ_1.

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with constants depending on ν and μ_1. From the representation

$$u(x,t) = u(x,0) + \int_0^t u_\xi(x,\xi)d\xi$$

we obtain the bound

$$\int_{\Omega_t} u^2(x,t) \leq 2 \int_\Omega u^2(x,0)dx + 2t \int_0^t y(t)dt.$$

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This provides

$$z(t) = \int_{\Omega_t} (u^2 + u_t^2 + a_{ij} u_{x_i} u_{x_j}) dx$$
$$\leq 2z(0) + (c + c_1 + 2t) \int_0^t z(t) dt + 2 \int_0^t \|f\|_{L_2(\Omega_t)} z^{\frac{1}{2}}(t) dt$$

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With the definition $\max_{0 \leq \xi \leq t} z(\xi) = \hat{z}(t)$ we have

$$\hat{z}(t) \leq 2z(0) + (c + c_1 + 2t) t \hat{z}(t) + 2\|f\|_{L_{2,1}(Q_t)} \hat{z}^{\frac{1}{2}}(t).$$

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For $t \in [0, t_1]$, where t_1 is defined by $(c + c_1 + 2t_1)t_1 = \frac{1}{2}$ we have

$$\hat{z}^{\frac{1}{2}}(t) \leq 4z^{\frac{1}{2}}(0) + 4\|f\|_{L_{2,1}(Q_t)}.$$

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For $t \in [t_1, 2t_1]$

$$\hat{z}^{\frac{1}{2}}(t) \leq 4z^{\frac{1}{2}}(t_1) + 4\|f\|_{L_{2,1}(Q_{t_1,t})}.$$

This implies for $t \in [0, T]$

$$\hat{z}^{\frac{1}{2}}(t) \leq c_2(t) z^{\frac{1}{2}}(0) + c_3(t)\|f\|_{L_{2,1}(Q_{t_1,t})},$$

where $c_2(t)$ and $c_3(t)$ are defined by the constants ν, μ_1 and t.

Generalized solution

Definition

We call $u(x,t)$ a generalized solution from the space $H^1(Q_T)$ if it belongs to $H_0^1(Q_T)$, satisfies $u|_{t=0} = \varphi(x)$ and the identity

$$\int_{Q_t} (-u_t \eta_t + a_{ij} u_{x_j} u_{x_i} + a_i u_{x_i} \eta + a u \eta) \, dx \, dt - \int_{\Omega} \psi \eta(x,0) \, dx = \int_{Q_t} f \eta \, dx \, dt$$

for all $\eta \in \hat{H}_0^1(Q_t)$.

- $H_0^1(Q_T) = \{u \in H^1(Q_T) : u|_{S_T} = 0\}$
- $\hat{H}_0^1(Q_T) = \{u \in H_0^1(Q_T) : u|_{t=T} = 0\}$

Uniqueness theorem in $H^1(Q_T)$

Theorem

Let the coefficients satisfy

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mu|\xi|^2, \quad \nu > 0,$$
$$\left|\frac{\partial a_{ij}}{\partial t}, \frac{\partial a_i}{\partial x_i}, a_i, a\right| \le \mu_1.$$

Then our problem cannot have more then one generalized solution in $H^1(Q_T)$.

Suppose the problem has two generalized solutions $u^{'}, u^{''}$.
$u = u' - u''$

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Consider particular test function

$$\eta(x,t) = \begin{cases} 0 & \text{for } b \leq t \leq T \\ \int_b^t u(x,\tau)\, d\tau & \text{for } 0 \leq t \leq b \end{cases}$$

$\rightsquigarrow \eta \in \hat{H}_0^1(Q_T)$, $\eta_t = u$ in $Q_b = \Omega \times (0,b)$

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$\rightsquigarrow \eta \in \hat{H}_0^1(Q_T)$, $\eta_t = u$ in $Q_b = \Omega \times (0,b)$
Using the representations $u_t = \eta_{tt}$, $u_{x_j} = \eta_{tx_j}$, $u = \eta_t$

$$\int_{Q_b} \eta_{tt}\eta_t - a_{ij}\eta_{tx_j}\eta_{x_i} - a_i\eta_{tx_i}\eta - a\eta_t\eta \, dx \, dt = 0$$

Integration by parts of first 3 terms

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega \eta_t^2(x,b) + a_{ij}\eta_{x_i}\eta_{x_j}|_{t=0}\ dx \\
&= -\int_{Q_b} \frac{1}{2}\frac{\partial a_{ij}}{\partial t}\eta_{x_i}\eta_{x_j} + a_i\eta_t\eta_{x_i} + (\frac{\partial a_i}{\partial x_i} - a)\eta_t\eta\ dx\ dt \\
&\leq c\int_{Q_b} |\eta_x|^2 + \eta_t^2 + \eta^2 dx\ dt,
\end{aligned}
$$

where c only depends on the constant μ_1.

$$\int_\Omega \eta_t^2(x,b) + a_{ij}\eta_{x_i}\eta_{x_j}|_{t=0}\ dx \le c\int_{Q_b} |\eta_x|^2 + \eta_t^2 + \eta^2 dx\ dt$$

Let us introduce

$$v(x,t) = \int_t^0 u(x,\tau)\ d\tau \quad \text{and} \quad v_i(x,t) = v_{x_i}(x,t) = \int_t^0 u_{x_i}(x,\tau)\ d\tau$$

Then we have

$$\eta_{x_i}(x,t) = v_i(x,b) - v_i(x,t),\ t \le b \quad \text{and} \quad \int_0^b \eta^2\ dt \le b^2 \int_0^b u^2\ d\tau.$$

$$\int_{\Omega} \eta_t^2(x,b) + a_{ij}\eta_{x_i}\eta_{x_j}|_{t=0}\; dx \le c \int_{Q_b} |\eta_x|^2 + \eta_t^2 + \eta^2 dx\; dt$$

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$$\begin{aligned} &\int_{\Omega} u^2(x,b) + a_{ij}(x,0)v_i(x,b)v_j(x,b)\; dx \\ &\le c\; \int_{Q_b} \sum_{i=1}^{n} (v_i(x,b) - v_i(x,t))^2 + (1+b^2)u^2\; dx\; dt \\ &\le 2cb \int_{\Omega} \sum_{i=1}^{n} v_i^2(x,b)\; dx + c \int_{Q_b} 2 \sum_{i=1}^{n} v_i^2 + (1+b^2)u^2\; dx\; dt \end{aligned}$$

$$\int_\Omega u^2(x,b) + (\nu - 2bc)\int_\Omega \sum_{i=1}^n v_i^2(x,b)\,dx \le c_1 \int_{Q_b} u^2 + \sum_{i=1}^n v_i^2 \,dx\,dt,$$

where $c_1 = c(2+b^2)$.

Lemma

Let $y(b)$ be non-negative, absolutely continuous on $[0,T]$ and for almost all $b \in [0,T]$ satisfy the inequality

$$y'(b) \le cy(b),$$

with a constant $c > 0$. Then

$$y(b) \le e^{cb} y(0).$$

$$y(b) = \int_{Q_b} u^2 + \sum_{i=1}^n v_i^2 \,dx\,dt = \int_0^b \int_\Omega u^2 + \sum_{i=1}^n v_i^2 \,dx\,dt$$

Existence theorem in $H^1(Q_T)$

Theorem

Let the coefficients satisfy

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \mu|\xi|^2, \quad \nu > 0,$$
$$\left|\frac{\partial a_{ij}}{\partial t}, a_i, a\right| \leq \mu_1.$$

Then the problem has a generalized solution from $H^1(Q_T)$ *for* $f \in L_{2,1}(Q_T)$, $\varphi \in H_0^1(\Omega)$, $\psi \in L_2(\Omega)$.

Let $\{\varphi_k(x)\}$ be a fundamental system in $H_0^1(\Omega)$ with $(\varphi_k, \varphi_l) = \delta_k^l$.
We look for $u^N = \sum_{k=1}^N c_k^N(t)\varphi_k(x)$ such that

$$
\begin{aligned}
&(u_{tt}^N, \varphi_l) + a(u^N, \varphi_l) = (f, \varphi_l), \quad l = 1, \dots, N \\
&{c_k^N}'(0) = (\psi, \varphi_k), \quad c_k^N(0) = \alpha_k^N,
\end{aligned}
$$

where

$$
a(u^N, \varphi_l) = (a_{ij} u_{x_j}^N, \varphi_{l x_i}) + (a_i u_{x_i}^N + a u^N, \varphi_l),
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and $\varphi^N(x) = \sum_{k=1}^N \alpha_k^N \varphi_k(x)$, which approximates φ in $\|.\|_{H^1(\Omega)}$ for $N \to \infty$.

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$$(u_{tt}^N, \varphi_l) + a(u^N, \varphi_l) = (f, \varphi_l), \quad l = 1, \ldots, N$$
$$c_k^{N\prime}(0) = (\psi, \varphi_k), \quad c_k^N(0) = \alpha_k^N,$$

where

$$a(u^N, \varphi_l) = (a_{ij}u_{x_j}^N, \varphi_{lx_i}) + (a_i u_{x_i}^N + au^N, \varphi_l),$$

and $\varphi^N(x) = \sum_{k=1}^N \alpha_k^N \varphi_k(x)$, which approximates φ in $\|.\|_{H^1(\Omega)}$ for $N \to \infty$.
System of linear ordinary differential equations of second order in t

$$\underline{c}''(t) = M^{-1}(\underline{f}(t) - K\underline{c}(t)),$$
$$\underline{c}'(0) = \underline{\psi}, \quad \underline{c}(0) = \underline{\alpha},$$

with $[M]_{ij} = (\varphi_j, \varphi_i)$, $[K]_{ij} = a(\varphi_j, \varphi_i)$ and $[\underline{f}]_i = (f, \varphi_i)$.
Picard-Lindelöf theorem $\rightsquigarrow$ unique solution $\underline{c}(t) \in W_1^2(0, T)$.

Multiply each of the equations by ${c_l^N}'(t)$ and sum up

$$(u_{tt}^N, u_t^N) + (a_{ij}u_{x_j}^N, u_{tx_i}^N) + (a_i u_{x_i}^N + a u^N, u_t^N) = (f, u_t^N).$$

For u^N the energy inequality holds, which provides

$$\int_{\Omega_t} (u^N)^2 + |u_x^N|^2 + (u_t^N)^2 dx \leq c,$$

where c independent of N and t.

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where c independent of N and t.
Therefore

$$\|u^N\|_{H^1(Q_T)} \leq c_1.$$

$H^1(Q_T)$ is reflexive $\Rightarrow$ subsequence $u^N \rightharpoonup u \in H^1(Q_T)$,
convergence is weakly in $H^1(Q_T)$ (u^N, u_t^N, $u_{x_i}^N$ conv. weakly in $L^2(Q_T)$)
and uniformly in $t \in [0, T]$ in the norm of $L^2(\Omega)$

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- initial condition $u|_{t=0} = \varphi(x)$
- Multiply each of the equations by $d_l(t) \in H^1(0, T)$, $d_l(T) = 0$, sum them up and integrate over $(0, T)$. We obtain

$$\int_{Q_T} u_{tt}^N \eta + a_{ij} u_{x_j}^N \eta_{x_i} + a_i u_{x_i}^N \eta + a u^N \eta \, dx \, dt = \int_{Q_T} f \eta \, dx \, dt$$

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  We denote by $\mathcal{M}_N$ the set of all such η.
  - For fixed η can take the limit along weakly convergent subsequence.

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We denote by $\mathcal{M}_N$ the set of all such η.

  - For fixed η can take the limit along weakly convergent subsequence.
  - $\bigcup_{N=1}^\infty \mathcal{M}_N$ dense in $\hat{H}^1(Q_T)$.