dG Space-Time Methods for Parabolic Problems

Daniel Jodlbauer

17.01.2017

Content

- Preliminaries
- dG space-time formulation
- Existence and uniqueness
- inf-sup condition
- Outlook: error estimates

Heat Equation

$$\partial_t u - \Delta u = f \quad \text{in } Q := \Omega \times (0, T) \tag{1}$$

$$u(x, t) = 0 \quad \text{in } \Sigma_D := \Gamma_D \times (0, T) \tag{2}$$

$$\nabla u \cdot n_x = g_N \quad \text{in } \Sigma_N := \Gamma_N \times (0, T) \tag{3}$$

$$u(x, 0) = u_0(x) \quad \text{in } \Sigma_0 := \Omega_0 \times \{0\} \tag{4}$$

dG Notations

On common interfaces Γ_{kl}:

$$\begin{aligned} v_k &:= v(x,t)\Big|_{\tau_k} \\ [\![v]\!] &:= v_k n_k + v_l n_l \\ [\![v]\!]_t &:= v_k n_k^t + v_l n_l^t \\ [\![v]\!]_x &:= v_k n_k^x + v_l n_l^x \\ \langle v \rangle &:= \frac{1}{2}(v_k + v_l) \end{aligned}$$

$$\{v\}^{\text{up}} := \begin{cases} v_k & \text{if } n_k^t > 0 \\ 0 & \text{if } n_k^t = 0 \\ v_l & \text{if } n_k^t < 0 \end{cases} \qquad \{v\}^{\text{down}} := \begin{cases} v_k & \text{if } n_k^t < 0 \\ 0 & \text{if } n_k^t = 0 \\ v_l & \text{if } n_k^t > 0 \end{cases}$$

dg Space-Time Formulation

Find $u_h \in S_h^p(\tau_N)$ such that $A(u_h, v_h) := a(u_h, v_h) + b(u_h, v_h) = \langle f, v_h \rangle_Q + \langle u_0, v_h \rangle_{\Sigma_0} + \langle g_N, v_h \rangle_{\Sigma_N}$ with

$$
\begin{aligned}
a(u_h, v_h) :=& \sum_{l=1}^{N} \int_{\tau_l} \nabla u_h \cdot \nabla v_h \, \mathrm{dx} \\
& - \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} \langle \nabla u_h \rangle \cdot [\![v_h]\!]_x \, \mathrm{ds} - \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} [\![u_h]\!]_x \cdot \langle \nabla v_h \rangle \, \mathrm{ds} \\
& + \sum_{\Gamma_{kl}} \frac{\sigma}{\bar{h}_{kl}} \int_{\Gamma_{kl}} [\![u_h]\!]_x \cdot [\![v_h]\!]_x \, \mathrm{ds} \\
b(u_h, v_h) :=& - \sum_{l=1}^{N} \int_{\tau_l} u_h \partial_t v_h \, \mathrm{dx} + \int_{\Sigma_T} u_h v_h \, \mathrm{ds} \\
& + \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} \{u_h\}^{\mathrm{up}} [\![v_h]\!]_t \, \mathrm{ds}
\end{aligned}
$$

Some Energy Norms

$$\|u\|_A^2 := \sum_{l=1} \|\nabla u\|_{L^2(\tau)}^2 + \sum_{\Gamma_{kl}} \frac{\sigma}{\bar{h}_{kl}} \|[\![u]\!]_x\|_{L^2(\Gamma_{kl})}^2$$

$$\|u\|_{A,*}^2 := \|u\|_A^2 + \sum_{\Gamma_{kl}} \bar{h}_{kl} \|\langle \nabla u \rangle\|_{L^2(\Gamma_{kl})}^2$$

$$\|u\|_B^2 := \sum_{l=1} h_l \|\partial_t u\|_{L^2(\tau)}^2 + \|u\|_{\Sigma_0}^2 + \|u\|_{\Sigma_T}^2 + \sum_{\Gamma_{kl}} \|[\![u]\!]_t\|_{L^2(\Gamma_{kl})}^2$$

$$\|u\|_{B,*}^2 := \sum_{l=1} h_l^{-1} \|u\|_{L^2(\tau)}^2 + \|u\|_{\Sigma_T}^2 + \sum_{\Gamma_{kl}} \|\{u\}^{\mathrm{up}}\|_{L^2(\Gamma_{kl})}^2$$

Existence & Uniqueness: Overview

1. Lower / Upper bound of a wrt. $\|\cdot\|_A$
2. Upper bound for b wrt. $\|\cdot\|_B$
3. Lower bound for b
4. Coercivity of A wrt. $\|\cdot\|_{\tilde{DG}}$
5. $\implies$ existence & uniqueness

For error estimates wrt. $\|\cdot\|_{DG}$:

6. Inf-sup for b wrt to $\|\cdot\|_B$
7. Boundedness and
8. inf-sup for A in the DG-norm

Inverse Inequalities and Other Relations

Assumptions:

- Quasi-uniformity
- Shape regular and local mesh grading: $h_l \sim h_k$

Lemma 1 (Inverse Inequalities)

$$\begin{aligned} \|v_h\|_{L^2(\Gamma_{kl})} &\leq c_I h^{-\frac{1}{2}} \|v_h\|_{L^2(\tau)} \\ \|\nabla v_h\|_{L^2(\Gamma_{kl})} &\leq c_I h^{-\frac{1}{2}} \|\nabla v_h\|_{L^2(\tau)} \\ \|v_h\|_{H^1(\Gamma_{kl})} &\leq c_I h^{-1} \|v_h\|_{L^2(\Gamma_{kl})} \\ \|v_h\|_{H^1(\tau)} &\leq c_I h^{-1} \|v_h\|_{L^2(\tau)} \end{aligned} \tag{5}$$

Lemma 2

$$\sum_{\Gamma_{kl}} h \|\langle \nabla u_h \rangle\|^2_{L^2(\Gamma_{kl})} \leq c_K \sum_{l=1}^{N} \|\nabla u_h\|^2_{L^2(\tau)}$$

First Boundedness & Coercivity Results

Lemma 3

We have:

$$a(u, v_h) \leq c_2^a \|u\|_{A,*} \|v_h\|_A,$$

$$b(u, v_h) \leq \|u\|_{B,*} \|v_h\|_B$$

and (if σ suff. large)

$$a(u_h, u_h) \geq \frac{1}{2} \|u_h\|_A^2$$

Proof.

Cauchy-Schwarz, lemma 2 and Young's inequality (for coercivity). □

Lower Bound for b

Lemma 4

$$\{u_h\}^{up} [\![u_h]\!]_t - \frac{1}{2}[\![u_h^2]\!]_t = \frac{1}{2}|n_k^t| \, [\![u_h]\!]^2$$

Lemma 5

$$b(u_h, u_h) \geq \frac{1}{2}\left[\|u\|_{\Sigma_0}^2 + \|u\|_{\Sigma_T}^2 + \sum_{\Gamma_{kl}} \|[\![u_h]\!]_t\|_{L^2(\Gamma_{kl})}^2\right]$$

Proof.

1/2-trick, Gauss' theorem and prev. lemma. □

Existence and Uniqueness

From the previous results we get

Theorem 6

$$A(u_h, u_h) \geq \frac{1}{2}\left[\|u_h\|_A^2 + \|u\|_{\Sigma_0}^2 + \|u\|_{\Sigma_T}^2 + \sum_{\Gamma_{kl}} \|[\![u_h]\!]_t\|_{L^2(\Gamma_{kl})}^2\right]$$
$$:= \frac{1}{2}\|u_h\|_{\tilde{DG}}^2 \tag{6}$$

- if $|\Gamma_D| > 0$, then $\|\cdot\|_{\tilde{DG}}$ is a norm
- $\implies$ existence and uniqueness of the discrete problem
- pure Neumann: $\forall v_h : \; A(u_h, v_h) = 0 \implies u_h = 0$

Inf-Sup Condition for b

Lemma 7

$b(u_h, u_h + \delta w_h) \geq c_1^b \|u_h\|_B^2,$ *with* $\delta := (c_g^{-1} 2 c_I^2 c_{R_2} c_g^3)^{-1}$

Lemma 8

$\|w_h\|_B \leq c_I^b \|u_h\|_B$

Theorem 9

$$\sup_{0 \neq v_h} \frac{b(u_h, v_h)}{\|v_h\|_B} \geq c_S^b \|u_h\|_B$$

Proof.

Using $v_h := u_h + \delta w_h$ with $w_h := h \partial_t u_h$, lemma 7 and 8 □

$b(u_h, v_h) := -\sum_{l=1}^N \int_{\tau_l} u_h \partial_t v_h \, \mathrm{dx} + \int_{\Sigma_T} u_h v_h \, \mathrm{ds} + \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} \{u_h\}^{\mathrm{up}} [\![v_h]\!]_t \, \mathrm{ds}$

$\|u\|_B^2 := \sum_{l=1} h \|\partial_t u\|_{L^2(\tau)}^2 + \|u\|_{\Sigma_0}^2 + \|u\|_{\Sigma_T}^2 + \sum_{\Gamma_{kl}} \|[\![u]\!]_t\|_{L^2(\Gamma_{kl})}^2$

Inf-Sup Condition for A

Lemma 10

$\|w_h\|_A \leq c_I^a \|u_h\|_A$

Theorem 11

If σ suff. large, then it holds that: $\sup\limits_{0 \neq v_h} \frac{A(u_h, v_h)}{\|v_h\|_{DG}} \geq c_S^A \|u_h\|_{DG}$

Theorem 12

$A(u, v_h) \leq c_2^A \|u\|_{DG,*} \|v_h\|_{DG}$, *for* $s > \frac{3}{2}$

Proof.

Using stability and boundedness estimates from before. □

$$\begin{aligned} \|u\|_{DG}^2 &:= \|u\|_A^2 + \|u\|_B^2 \\ \|u\|_{DG,*}^2 &:= \|u\|_{A,*}^2 + \|u\|_{B,*}^2 \end{aligned} \tag{7}$$

$\|u\|_A^2 := \sum_{l=1} \|\nabla u\|_{L^2(\tau)}^2 + \sum_{\Gamma_{kl}} \frac{\sigma}{h_{kl}} \|[\![u]\!]_x\|_{L^2(\Gamma_{kl})}^2$

Error Estimates

Theorem 13

Let τ_N be quasi-uniform, $u \in H^s, s \geq 2$, σ sufficiently large and u_h the solution of the discrete problem. Then we have:

$$\|u - u_h\|_{DG} \leq ch^{\min\{s,p+1\}-1} |u|_{H^s(\tau_N)}$$

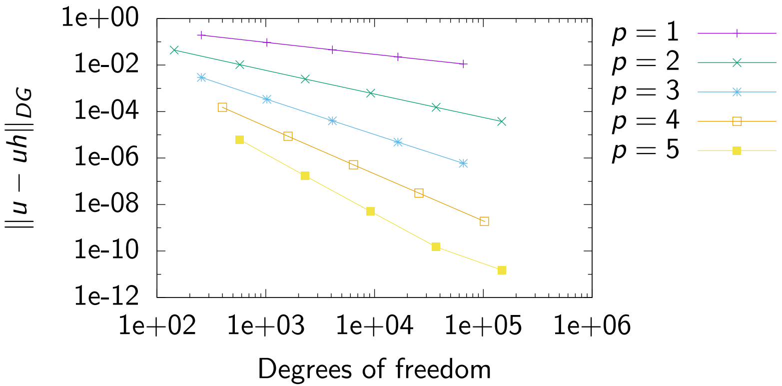
Proof.

M. Neumüller, *Space-Time Methods : Fast Solvers and Applications*,

□

Numerical Results

$$u(x,t) := cos(\pi t) sin(\pi x)$$



Thank you for your attention!