A overview of discretizations for space time methods

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Overview

- Space Time-Discretizations
  1. M. Neumüller
  2. O. Steinbach
  3. C. Mollet
  4. C. Schwab/R. Stevenson and R. Andreev

Strategies to discetize space time formulations

- All presented formulation are discretizations based on $L_2(0, T; V)$, $H^1(0, T; V^*)$
- No formulation based on Ladyzhenskaya.
- Continuous Galerkin vs. discontinuous Galerkin
  - cG: Steinbach, Schwab/Stevenson, Andreev, Mollet
  - dG: Neumüller
- Square matrix vs. rectangular matrix
  - Square matrix: Steinbach, Neumüller
  - Rectangular matrix: Schwab/Stevenson, Andreev, Mollet

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M. Neumüller

Find $u_h \in S_h^p(\tau_N)$:

$$A(u_h, v_h) = \langle f, v_h \rangle_Q + \langle u_0, v_h \rangle_{\Sigma_0} + \langle g_N, v_h \rangle_{\Sigma_N} \quad \forall v_h \in S_h^p(\tau_N),$$

where

$$\begin{aligned}
A(u_h, v_h) :=& a(u_h, v_h) + b(u_h, v_h) \\
a(u_h, v_h) :=& \sum_{l=1}^{N} \int_{\tau_l} \nabla u_h \cdot \nabla v_h dx - \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} \langle \nabla u_h \rangle \cdot [\![v_h]\!]_x + \langle \nabla v_h \rangle \cdot [\![u_h]\!]_x ds \\
&+ \sum_{\Gamma_{kl}} \frac{\sigma}{h_{kl}} \int_{\Gamma_{kl}} [\![u_h]\!]_x \cdot [\![v_h]\!]_x ds \\
b(u_h, v_h) :=& -\sum_{l=1}^{N} \int_{\tau_l} u_h \partial_t v_h dx + \int_{\Sigma_T} u_h v_h ds + \sum_{\Gamma_{kl}} \int_{\Gamma_{kl}} \{u_h\}^{up} [\![v_h]\!]_t ds
\end{aligned}$$

Properties

- Existence and Uniqueness:

$$A(u_h, v_h) \geq \frac{1}{2} \|u_h\|^2_{\tilde{DG}}.$$

- Discrete inf-sup condition for A:

$$\sup_{v_h} \frac{A(u_h, v_h)}{\|v_h\|_{DG}} \geq c\|u_h\|_{DG}.$$

- A-priori error estimate:

$$\|u - u_h\|_{DG} \leq ch^{\min\{s,p+1\}-1} |u|_{H^s(\tau_N)},$$

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O. Steinbach

$X_h \subset X$, $Y_h \subset Y$, where $X_h \subset Y_h$.
Find $\overline{u}_h \in X_h$:

$$b(\overline{u}_h, v_h) = \langle f, v_h \rangle_Q - b(\overline{u}_0, v_h) \quad \forall v_h \in Y_h.$$

- X contains the condition: $u(x,0) = 0$.
- No condition build in Y.
- $X_h := P_1(Q_h) \cap X$.
- $Y_h := P_1(Q_h) \cap Y$.
- In the numerical realization: $Y_h := X_h$, i.e., neglecting basis functions w.r.t. the zero initial conditions. $\rightsquigarrow$ square matrix.

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Properties

- Discrete stability

$$\sup_{v_h \in Y_h} \frac{b(u_h, v_h)}{\|v_h\|_{L_2(0,T;H_0^1(\Omega))}} \geq \frac{1}{2\sqrt{2}} \|u_h\|_{X_h} \quad u_h \in X_h$$

- A-priori error estimate

$$\|\overline{u} - \overline{u}_h\|_{L_2(0,T;H_0^1(\Omega))} \leq c h^{s-1} |\overline{u}|_{H^s(Q)}, \quad s \in [1, p+1]$$

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C. Mollet

$S_j \subset X$ and $Q_l \subset Y$.

$$u_j := \arg\min_{v \in S_j} \sup_{q_l \in Q_l} \frac{|\langle Bv_j - f, q_l \rangle|}{\|q_l\|_Y}$$

- The testspace is larger than the solution space: $l \geq j + L$.
- Smoother spaces: $X'_+ \hookrightarrow X'$, $Y_+ \hookrightarrow Y$
- Choose $Q_l \subset Y_+$ and let $\tilde{S}_l \subset X'_+$
- It is sufficient to consider $\tilde{S}_l := S_l$

- Assumption 1: (regularity) $(B')^{-1} \in \mathcal{L}(X'_+, X_+)$
- Assumption 2a: (inverse inequality)

$$\|\tilde{v}_j\|_{X_+} \leq C\rho^j \|\tilde{v}_j\|_{X'}, \quad \forall \tilde{v}_j \in \tilde{S}_j$$

- Assumption 2b: (approximation estimate)

$$\inf_{q_l \in Q_l} \|q - q_l\|_Y \leq C\rho^{-j} \|q\|_{Y_+}, \quad \forall q \in \tilde{Y}_+$$

- Assumption 2c: (reverse Cauchy inequality)

$$\forall v_j \in S_j \exists \tilde{v}_j \in \tilde{S}_j : \|v_j\|_X \|\tilde{v}_j\|_{X'} \leq C\langle v_j, \tilde{v}_j\rangle_{X \times X'}.$$

Discrete inf-sup follows with $l \geq j + L$.

Special setup

- $S_j := S_j^t \otimes S_j^x$, where $S_j^x \subset H^m(\Omega)$, $S_j^t \subset L_2(0,T)$
- $Q_l := Q_l^t \otimes Q_l^x$, where $Q_l^x \subset H^{2m}(\Omega)$, $Q_l^t \subset H^1_{\{T\}}(0,T)$
- Sequences $\{S_j^x\}_{j=j_0}^\infty, \{S_j^t\}_{j=j_0}^\infty, \{Q_l^x\}_{l=l_0}^\infty, \{Q_l^t\}_{l=l_0}^\infty$ must be nested

$$S_{j_0}^x \subset S_{j_1}^x \subset \ldots \subset S_j^x \subset \ldots \subset H^m(\Omega)$$

- $\tilde{S}_j$ only needed for the analysis of the discrete inf-sup condition.

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- $X_h := E \otimes V_h \subset X$ and $Y_h := (F \otimes V_h) \times V_h \subset Y$
- Finite dim. subspaces: $E \subset H^1(0,T)$, $F \subset L_2(0,T)$, $V_h \subset V$.
- Discrete variational formulation: Find $u_h \in X_h$:

$$B(u_h, v_h) = b(v_h), \quad \forall v_h \in Y_h$$

- Two types of temporal subspaces:
  Let τ_E, τ_F be meshes in E, F.
  1. $E \ldots P_1$-elements on τ_E, $F \ldots P_0$-elements on $\tau_F := \tau_E$
  2. E as in 1., τ_F be a uniform refinement of τ_E.
     $F \ldots P_0$-elements on τ_F
- Type 1 $\rightsquigarrow$ continuous Galerkin time-stepping scheme.
- Type 2 $\rightsquigarrow$ Petrov-Galerkin, $\dim(Y_h) > \dim(X_h)$.
  (overdetermined system)

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Stability of Type 2

- Introduce norms $|||\cdot|||_X$ and $|||\cdot|||_Y$ on X_h, Y_h
  - induced by $M : X_h \to X'$ and $N : Y_h \to Y'$.
  - equvalent to $\|\cdot\|_X$ and $\|\cdot\|_Y$
- Procedure similar to Mollet:

$$u_h := \arg\min_{v \in X_h} \sup_{q_l \in Y_l} \frac{|B(v_j, q_l) - b(q_l)|}{|||q_l|||_Y}$$

- Discrete inf-sup condition: For any $V_h \subset V, \dim(V_h) < \infty$

$$\inf_{u_h \in X_h} \sup_{v_h \in Y_h} \frac{B(u_h, v_h)}{\|u_h\|_X \|v_h\|_Y} \geq C_h > 0$$

- Existence of a unique solution $u_h \in X_h$
- Estimation by best approximation

$$\|u - u_h\|_X \leq C_h \inf_{w_h \in X_h} \|u - w_h\|_X$$

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Matrix representation

- Let Φ, Ψ be a basis of X_h, Y_h

$$\boldsymbol{B} := B(\Phi, \Psi),\ \boldsymbol{b} := b(\Phi, \Psi),\ \boldsymbol{M} := (M\Phi)(\Phi),\ \boldsymbol{N} := (N\Psi)(\Psi)$$

- $\boldsymbol{u} = \arg\min_{\boldsymbol{w}} \|\boldsymbol{B}\boldsymbol{w} - \boldsymbol{b}\|_{\boldsymbol{N}^{-1}}$ is equivalent to

$$\boldsymbol{B}^T\boldsymbol{N}^{-1}\boldsymbol{B}\boldsymbol{u} = \boldsymbol{B}^T\boldsymbol{N}^{-1}\boldsymbol{b}$$

- $\boldsymbol{M}$ and $\boldsymbol{B}^T\boldsymbol{N}^{-1}\boldsymbol{B}$ spectrally equivalent (preconditioning)
- Choice of M, N (Riesz mappings), $w \in X, v = (v_1, v_2) \in Y$.
  - $(Mw)(w) := \|w\|_X^2 = \|w\|_{L^2(0,T,V)}^2 + \|\partial_t w\|_{L^2(0,T,V')}^2$
  - $(Nv)(v) := \|v\|_Y^2 = \|v_1\|_{L^2(0,T,V)}^2 + \|v_2\|_H^2$