Stability of Petrov-Galerkin discretizations: Application to the space-time weak formulation for parabolic evolution problems

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Schedule for the talks

Sahar Faghfouri

- Introduction to the Petrov-Galerkin discretization
- General Set-up and Main Result

Felix Scholz

- Application to Space-Time Weak Formulation

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Our goal

We want to prove that

- under standard approximation and smoothness conditions
- without any further coupling between the discrete trial and test spaces for sufficiently regular operators

the **discrete inf-sup condition** for a priori fixed **Petrov-Galerkin discretization** is satisfied uniformly .

Outline

Basic Concepts

Let $\mathrm{B} \in \mathcal{L}(\mathrm{X}, \mathrm{Y}')$, X and Y' be two Hilbert spaces.

Generic operator equation:

$$\mathrm{Bu} = \mathrm{f} \qquad \mathrm{u} \in \mathrm{X}, \qquad \mathrm{f} \in \mathrm{Y}'$$

Boundedly invertible operator

Three conditions:

(i) *Boundedness* : $$\mathrm{C_B} := \sup_{\mathrm{v}\in \mathrm{X}\setminus\{0\}} \sup_{\mathrm{q}\in \mathrm{Y}\setminus\{0\}} \frac{|\langle \mathrm{Bv},\mathrm{q}\rangle|}{\|\mathrm{v}\|_{\mathrm{X}}\|\mathrm{q}\|_{\mathrm{Y}}} < \infty$$

(ii) *Inf-sup condition* : $$\mathrm{c_B} := \inf_{\mathrm{v}\in \mathrm{X}\setminus\{0\}} \sup_{\mathrm{q}\in \mathrm{Y}\setminus\{0\}} \frac{|\langle \mathrm{Bv},\mathrm{q}\rangle|}{\|\mathrm{v}\|_{\mathrm{X}}\|\mathrm{q}\|_{\mathrm{Y}}} > 0$$

(iii) *Surjectivity* : $$\sup_{\mathrm{v}\in \mathrm{X}\setminus\{0\}} |\langle \mathrm{Bv},\mathrm{q}\rangle| > 0 \quad \forall \mathrm{q} \in \mathrm{Y}\setminus\{0\}$$

Babuska Aziz theorem

The operator B is **boundedly invertible** if and only if the operator is **bounded**, **surjective** and if an **inf-sup condition** is fulfilled.

Petrov-Galerkin discretization

Petrov-Galerkin discretization w.r.t discrete subspaces $S_j \subset X$ and $Q_l \subset Y$:

The Petrov-Galerkin solution $u_j \in S_j$ is given by the solution of the variational problem

$$\langle Bu_j, q_l \rangle = \langle f, q_l \rangle \qquad \text{for all } q_l \in Q_l.$$

The **minimal residual Petrov-Galerkin** solution of $Bu = f$ is defined as the minimizer of the functional residual

$$u_j := \arg\min_{v_j \in S_j} \sup_{q_l \in Q_l \setminus \{0\}} \frac{|\langle Bv_j - f, q_l \rangle|}{\|q_l\|_Y}$$

Three conditions for discrete subspaces

The operator B is bounded on the discrete subspaces as well since

$$\sup_{\mathrm{v_j} \in \mathrm{S_j} \setminus \{0\}} \sup_{\mathrm{q_l} \in \mathrm{Q_l} \setminus \{0\}} \frac{|\langle \mathrm{B v_j}, \mathrm{q_l} \rangle|}{\|\mathrm{v_j}\|_{\mathrm{X}} \|\mathrm{q_l}\|_{\mathrm{Y}}} \leq \sup_{\mathrm{v} \in \mathrm{X} \setminus \{0\}} \sup_{\mathrm{q} \in \mathrm{Y} \setminus \{0\}} \frac{|\langle \mathrm{B v}, \mathrm{q} \rangle|}{\|\mathrm{v}\|_{\mathrm{X}} \|\mathrm{q}\|_{\mathrm{Y}}} < \infty.$$

But for general B, the inf-sup condition on the spaces X and Y does generally not imply its discrete counterpart

$$\inf_{\mathrm{v_j} \in \mathrm{S_j} \setminus \{0\}} \sup_{\mathrm{q_l} \in \mathrm{Q_l} \setminus \{0\}} \frac{|\langle \mathrm{B v_j}, \mathrm{q_l} \rangle|}{\|\mathrm{v_j}\|_{\mathrm{X}} \|\mathrm{q_l}\|_{\mathrm{Y}}} =: \beta_{j,l} > 0$$

Remark on discrete inf-sup condition

- The discrete inf-sup condition determines the stability of a Petrov-Galerkin approach
- The constants $\beta_{j,l}$ can be bounded uniformly by a constants $\beta > 0$
- Otherwise stability problem appears for the limit $\mathrm{S_j} \to \mathrm{X}$ and $\mathrm{Q_l} \to \mathrm{Y}$

Quasi optimality of Petrov-Galerkin solutions

Theorem

Let $B \in \mathcal{L}(X, Y')$ *and assume that the discrete inf-sup condition with respect to* $S_j \subset X$ *and* $Q_l \subset Y$ *is satisfied. Then for any* $u \in X$ *there exists a unique* $u_j \in S_j$ *which satisfies*

$$\mathcal{R}_l(u_j) = \inf_{v_j \in S_j} \mathcal{R}_l(v_j),$$

$$\mathcal{R}_l(v_j) := \sup_{q_l \in Q_l \setminus \{0\}} \frac{|\langle Bv_j - Bu, q_l \rangle|}{\|q_l\|_Y}.$$

Moreover, there holds the quasi-optimality estimate

$$\|u - u_j\|_X \leq \frac{C_B}{\beta_{j,l}} \inf_{v_j \in S_j} \|u - v_j\|_X,$$

with discrete inf-sup constant $\beta_{j,l}$ *and continuity constant* C_B *.*

Quasi optimality of Petrov-Galerkin solutions

If the discrete inf-sup condition is satisfied uniformly ,i.e., $\inf_{j,l} \beta_{j,l} \geq \beta > 0$ for some β independent of j and l, then $\mathrm{u_j}$ is the quasi-optimal approximation of u with

$$\|\mathrm{u} - \mathrm{u_j}\|_{\mathrm{X}} \leq \frac{\mathrm{C_B}}{\beta} \inf_{\mathrm{v_j} \in \mathrm{S_j}} \|\mathrm{u} - \mathrm{v_j}\|_{\mathrm{X}}.$$

otherwise, if $\beta_{j,l}$ tends to zero for $j, l \to \infty$, the quasi-optimality constants $\frac{\mathrm{C_B}}{\beta_{j,l}}$ tend to infinity.

General Set-up and Main Result

Let X, Y and $\mathcal{H}$ are three Hilbert spaces. By identifying the pivot space $\mathcal{H}$ with its dual we obtain the Gelfand triple

$$\mathrm{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathrm{X}'.$$

Let $\mathrm{X}'_+ \hookrightarrow \mathrm{X}'$ and $\mathrm{Y}_+ \hookrightarrow \mathrm{Y}$ are Hilbert spaces which are continuously and densely embedded in X' and Y.

General Set-up and Main Result

Assuming a B.I.O. $\mathrm{B} \in \mathcal{L}(\mathrm{X}, \mathrm{Y}')$, there are constants $0 < \mathrm{C_B} < \infty$ and $0 < \mathrm{c_B} < \infty$ with

$$\|\mathrm{Bv}\|_{\mathrm{Y}'} \leq \mathrm{C_B}\|\mathrm{v}\|_X \quad (\mathrm{v} \in \mathrm{X}), \text{ and } \quad \|\mathrm{B}^{-1}\tilde{\mathrm{q}}\|_{\mathrm{X}} \leq \mathrm{c_B^{-1}}\|\tilde{\mathrm{q}}\|_{\mathrm{Y}'} \quad (\tilde{\mathrm{q}} \in \mathrm{Y}').$$

To prove a U.D. inf-sup condition the dual operator $\mathrm{B}' : \mathrm{Y} \to \mathrm{X}'$ by

$$\langle \mathrm{v}, \mathrm{B}'\mathrm{q}\rangle_{\mathrm{X}\times\mathrm{X}'} := \langle \mathrm{Bv}, \mathrm{q}\rangle_{\mathrm{Y}'\times\mathrm{Y}} \quad (\mathrm{v} \in \mathrm{X},\ \mathrm{q} \in \mathrm{Y})$$

is defined. It is well known that $\mathrm{B} \in \mathcal{L}(\mathrm{X}, \mathrm{Y}')$ implies $\mathrm{B}' \in \mathcal{L}(\mathrm{Y}, \mathrm{X}')$ with

$$\|\mathrm{B}'\|_{\mathrm{Y}\to\mathrm{X}'} = \|\mathrm{B}\|_{\mathrm{X}\to\mathrm{Y}'} \quad \text{and} \quad \|(\mathrm{B}'^{-1})\|_{\mathrm{X}'\to\mathrm{Y}} = \|\mathrm{B}^{-1}\|_{\mathrm{Y}'\to\mathrm{X}}.$$

Regularity assumption on operator B

Assumption 1

Assume that the dual operator B' satisfies the regularity condition

$$(\mathrm{B}')^{-1} \in \mathcal{L}(\mathrm{X}'_+, \mathrm{Y}_+), \quad \text{with} \quad \|(\mathrm{B}')^{-1}\|_{\mathcal{L}(\mathrm{X}'_+, \mathrm{Y}_+)} \leq \mathrm{C}_+,$$

with a constant $0 < \mathrm{C}_+ < \infty$.

Jackson-Bernstein estimates

Assumption 2 (part I)

Let $\{\mathrm{S_j}\}_{\mathrm{j=j_0}}^{\infty} \subset \mathrm{X}$, and $\{\tilde{\mathrm{S}}_\mathrm{j}\}_{\mathrm{j=j_0}}^{\infty} \subset \mathrm{X}'_+$ and $\{\mathrm{Q_l}\}_{\mathrm{l=l_0}}^{\infty} \subset \mathrm{Y}_+$ be closed subspaces such that for some $\rho > 1$ the **Bernstein estimate**

$$\|\tilde{\mathrm{v}}_\mathrm{j}\|_{\mathrm{X}'_+} \leq \mathrm{C_{B,X'}} \rho^j \|\tilde{\mathrm{v}}_\mathrm{j}\|_{\mathrm{X}'} \quad \tilde{\mathrm{v}}_\mathrm{j} \in \tilde{\mathrm{S}}_\mathrm{j},$$

as well as the **Jackson estimate**

$$\inf_{\mathrm{q_l} \in Q_l} \|\mathrm{q} - \mathrm{q_l}\|_\mathrm{Y} \leq \mathrm{C_{J,Y}} \rho^{-l} \|\mathrm{q}\|_{\mathrm{Y}_+}, \quad \mathrm{q} \in \mathrm{Y}_+$$

are satisfied with constants $\mathrm{C_{B,X'}}, \mathrm{C_{J,Y}} > 0$.

Jackson-Bernstein estimates

Assumption 2 (part II)

Moreover, we assume that the reverse Cauchy-Schwarz inequality holds on X:

For every $\mathrm{v_j} \in \mathrm{S_j}$ there exists an element $\tilde{\mathrm{v}}_\mathrm{j} \in \tilde{\mathrm{S}}_\mathrm{j}$, depending on $\mathrm{v_j}$, such that

$$\|\mathrm{v_j}\|_\mathrm{X} \|\tilde{\mathrm{v}}_\mathrm{j}\|_{\mathrm{X}'} \leq \mathrm{C_{CS}} \langle \mathrm{v_j}, \tilde{\mathrm{v}}_\mathrm{j} \rangle_{\mathrm{X}\times\mathrm{X}'},$$

with a constant $\mathrm{C_{CS}} > 0$, where $\langle .,. \rangle_{\mathrm{X}\times\mathrm{X}'}$ denotes the duality pairing between X and X' induced by the pivot space $\mathcal{H}$.

Lemma

Assume that Assumption 1 and Assumption 2 are fulfilled. Then, for arbitrary $\tilde{v}_j \in \tilde{S}_j$ there exists an element $q_l \in Q_l$, depending on $\tilde{v}_j$ such that

$$\|\tilde{v}_j - B' q_l\|_{X'} \leq C_{J,X'} \rho^{-(l-j)} \|\tilde{v}_j\|_{X'}$$

with a constant $C_{J,X'} := C_B C_+ C_{J,Y} C_{B,X'}$ and

$$\|q_l\|_Y \leq c_B^{-1} (C_{J,X'} \rho^{-(l-j)} + 1) \|\tilde{v}_j\|_{X'}$$

Theorem

Assume that Assumptions 1 and 2 are fulfilled. Choose $L \in \mathbb{N}$ *such that*

$$C_{CS} C_{J,X'} \rho^{-L} < 1$$

with constants $C_{J,X'}$ *and* C_{CS} *as in Lemma and reverse Cauchy-Schwarz inequality and set*

$$l \geq j + L$$

for any refinement level j. *Then the discrete inf-sup condition*

$$\inf_{v_j \in S_j \setminus \{0\}} \sup_{q_l \in Q_l \setminus \{0\}} \frac{|\langle B v_j, q_l \rangle|}{\|v_j\|_X \|q_l\|_Y} \geq \beta > 0$$

is satisfied with a constant β *uniformly bounded away from* 0 *as* $j \to \infty$. *In particular,* β *is given by*

$$\beta := \frac{C_{CS}^{-1} - C_{J,X'} \rho^{-L}}{c_B^{-1}(C_{J,X'} \rho^{-L} + 1)}.$$

Thank you for your attention