A Space-Time Petrov-Galerkin method for linear wave equations

Sahar Faghfouri

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Schedule for the talks

Sahar Faghfouri

- Linear wave equations
- A space-time setting for linear hyperbolic operations

Felix Scholz

- Discontinuous Galerkin methods for linear systems of conservation laws
- A Petrov-Galerkin space-time discretization

Bernhard Endtmayer

- Duality based goal-oriented error estimation

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Outline

Introduction

- A hyperbolic PDE of order n is a PDE that has a well-posed initial value problem for the first $n-1$ derivatives.
- Many equations in mechanics are hyperbolic.
- A linear hyperbolic equation in one dimensional is called wave equation.

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- A linear hyperbolic equation in one dimensional is called wave equation.

A simple 1D wave

Wave equation

The simplest equation expressing a wave is given by

$$\partial_t^2 \varphi(t,x) = c^2 \partial_x^2 \varphi(t,x)$$

where $t \in (0,T)$.

$t = 0 \dots$ Initial time

$T > 0 \dots$ Final time

$x \in \mathbb{R} \dots$ Position on the real line

$\varphi(t,x) \dots$ Displacement at t and position x

$c > 0 \dots$ Wave speed

A simple 1D wave

For given initial displacement $\varphi(0,.)$ and velocity $\partial_t\varphi(0,.)$, the solution is obtained by d'Alembert formula:

$$\varphi(t,x) = \frac{1}{2}\left(\varphi(0,x-ct) + \varphi(0,x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \partial_t\varphi(0,\xi)d\xi\right)$$

Solutions in the bounded interval $\Omega = (0,\pi)$ with Dirichlet boundary conditions: $\varphi(t,0) = \varphi(t,\pi) = 0$ can be expand into eigenmodes of the operator $A\varphi = -\partial_x^2\varphi$ in the domain $\mathcal{D}(A) = \mathrm{H}_0^1(\Omega)$:

$$\varphi(t,x) = \sum_{k=1}^{\infty}\Big(\alpha_k cos(ckt) + \beta_k sin(ckt)\Big) sin(kx)$$

Coefficients are determined by the initial displacement $\varphi(0,\cdot)$ and velocity $\partial_t\varphi(0,\cdot)$.

A simple 1D wave

Special case $\varphi(0,x) = 1$ and $\partial_t \varphi(0,x) = 0$ for $x \in (0,\pi)$ and $c = 1$

Fourier representation:

$$\begin{aligned}\varphi(t,x) &= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}cos((2k+1)t)sin((2k+1)x)\\ &= \frac{1}{2}(\varphi_0(x+t)+\varphi_0(x-t))\end{aligned}$$

with the periodic function:

$$\varphi_0(x) = \begin{cases} 1 & x \in (0,\pi)+2\pi\mathbb{Z} \\ 0 & x \in \pi\mathbb{Z} \\ -1 & x \in (-\pi,0)+2\pi\mathbb{Z} \end{cases}$$

Waves in solids

Stress rate

$$\partial_t \sigma = \partial_t(\hat{\Sigma}(F)) = \mathrm{D}(\hat{\Sigma}(\mathrm{D}\varphi))\partial_t(F) = \mathrm{D}(\hat{\Sigma}(\mathrm{D}\varphi))(\mathrm{Dv}) \quad (*)$$

$\varphi(\cdot,\cdot)\dots$ Deformation vector in elastic solid $\Omega \subset \mathbb{R}^3$

$\mathrm{v} = \partial_t \varphi \dots$ Velocity

$\sigma = \hat{\Sigma}(F)\dots$ Stress tensor

$\hat{\Sigma}(\cdot)\dots$ Stress response function

$F = \mathrm{D}\varphi = \nabla\varphi \dots$ Deformation gradient

Linear approximation of $(*)$ by assuming small strains and $\varphi \approx id$:

$$\partial_t \sigma = \mathrm{C}[\mathrm{Dv}]$$

$\mathrm{C} = \mathrm{D}\hat{\Sigma}(\mathrm{I})\dots$ Elasticity tensor

Waves in solids

The elasticity tensor

$$\sigma = \lambda \, tr(\varepsilon)\mathrm{I} + 2\mu\varepsilon = \mathrm{C}\varepsilon(\mathrm{u})$$

in isotropic media is characterized by Lame parameters $\lambda \geq 0$ and $\mu > 0$.

$\varphi = \mathrm{x} + \mathrm{u} \cdots$ Deformation vector in elastic solid $\Omega \subset \mathbb{R}^3$

$\mathrm{v} = \partial_t \varphi = \partial_t \mathrm{u} \cdots$ Velocity

$\sigma = \mathrm{C}\varepsilon(\mathrm{u}) \cdots$ Stress tensor

So the stress rate for linear elasticity

$$\partial_t \sigma = \partial_t(\mathrm{C}\varepsilon(\mathrm{u})) = \mathrm{C}\varepsilon(\partial_t \mathrm{u}) = \mathrm{C}\varepsilon(\mathrm{v})$$

Waves in solids

Introducing the compression modulus $\kappa = \dfrac{2\mu + 3\lambda}{3}$ and the shear term

$$dev(\sigma) = \sigma - \frac{1}{3} trace(\sigma)\mathrm{I}$$

Implies

$$\mathrm{C}\varepsilon = 2\mu\varepsilon + \lambda\, trace(\varepsilon)\mathrm{I} = 2\mu\, dev(\varepsilon) + \kappa\, trace(\varepsilon)\mathrm{I}$$

and

$$\mathrm{C}^{-1}\sigma = \frac{1}{2\mu} dev(\sigma) + \frac{3}{\kappa} trace(\sigma)\mathrm{I}$$

Waves in solids

Newton's law for the balance of forces

The equation

$$\rho \partial_t \mathrm{v} = \operatorname{div} \sigma + \mathrm{b}$$

$\rho \ldots$ Mass density

$\partial_t \mathrm{v} \ldots$ Acceleration

$\mathrm{b} \ldots$ Body force

Waves in compressible fluids

Compressible flow

Compressible flow is the branch of fluid mechanics that deals with flows having significant changes in fluid density.

In fluids shear forces can be neglected i.e we assume $\mu \longrightarrow 0$. Then stress

$$\sigma = \mathrm{pI}$$

is isotropic with hydrostatic pressure

$$\mathrm{p} = \frac{1}{3} \mathit{trace}\ \sigma$$

and compression waves are described by

$$\partial_t \mathrm{p} = \kappa \operatorname{div} \mathrm{v}, \quad \rho \partial_t \mathrm{v} = \nabla \mathrm{p} + \mathrm{b}$$

$$\partial_t^2 \mathrm{p} - \mathrm{C} \Delta \mathrm{p} = \mathrm{f}$$

Electro-magnetic waves

Definition

Electromagnetical fields are described by Maxwell's equations introduced by "James clerk Maxwell" in 1862. The electromagnetic quantities involved in Maxwell's equations depend $x \in \Omega \subseteq \mathbb{R}^d$, $t \in [0, T]$.

Maxwell system

$$\begin{aligned}
\partial_t \mathrm{D} - \mathrm{curl}\mathrm{H} &= -\mathrm{J} \quad \texttt{Ampere's law} \\
\mathrm{div}\mathrm{D} &= \rho \quad \texttt{Gauss law} \\
\partial_t \mathrm{B} + \mathrm{curl}\mathrm{E} &= 0 \quad \texttt{Faraday's law} \\
\mathrm{div}\mathrm{B} &= 0 \quad \texttt{Gauss law}
\end{aligned}$$

D:electric displacement	H:magnetic field	J:electric current density
ρ:electric charge density	B:magnetic field induction	E:electric field

Electro-magnetic waves

Vacuum

In vacuum with no electric charges we have

$$\mathrm{J} = 0, \quad \rho = 0$$

and the material laws

$$\mathrm{D} = \varepsilon \mathrm{E}$$
$$\mathrm{B} = \mu \mathrm{H}$$

with

$$\text{permeability } \mu > 0$$
$$\text{permittivity } \varepsilon > 0$$

First-order differential systems

For all discussed cases, we obtain a system of J equations in $\mathbb{R}^{\mathrm{D}}$

$$\mathrm{M}\partial_t \mathrm{u} + \mathrm{A}\mathrm{u} = \mathrm{f}$$

$\mathrm{A}\ldots$ First differential operator $\quad$ $\mathrm{M}\ldots$ Weighting operator

- Elastic waves

$$\mathrm{u} = (\sigma, \mathrm{v}), \quad \mathrm{A}(\sigma, \mathrm{v}) = -(\varepsilon(\mathrm{v}), \mathrm{div}\sigma), \quad \mathrm{M}(\sigma, \mathrm{v}) = (\mathrm{C}^{-1}\sigma, \rho\mathrm{v})$$

$$\begin{cases} \mathrm{C}^{-1}\partial_t \sigma - \varepsilon(\mathrm{v}) = \mathrm{f}_1 \\ \rho\partial_t \mathrm{v} - \mathrm{div}\sigma = \mathrm{f}_2 \end{cases}$$

First-order differential systems

$$\mathrm{M}\partial_t \mathrm{u} + \mathrm{A}\mathrm{u} = \mathrm{f}$$

- Acoustic waves

$$\mathrm{u} = (\mathrm{p}, \mathrm{v}), \quad \mathrm{A}(\mathrm{p}, \mathrm{v}) = -(\mathrm{div}\mathrm{v}, \nabla \mathrm{p}), \quad \mathrm{M}(\mathrm{p}, \mathrm{v}) = (\kappa^{-1}\mathrm{p}, \rho \mathrm{v})$$

$$\begin{cases} \kappa^{-1}\partial_t \mathrm{p} - \mathrm{div}\mathrm{v} = \mathrm{f}_1 \\ \rho\partial_t \mathrm{v} - \nabla \mathrm{p} = \mathrm{f}_2 \end{cases}$$

- Electro-magnetic waves

$$\mathrm{u} = (\mathrm{H}, \mathrm{E}), \quad \mathrm{A}(\mathrm{H}, \mathrm{E}) = (\mathrm{curl}\mathrm{E}, -\mathrm{curl}\mathrm{H}), \quad \mathrm{M}(\mathrm{H}, \mathrm{E}) = (\mu \mathrm{H}, \varepsilon \mathrm{E})$$

$$\begin{cases} \mu\partial_t \mathrm{H} + \mathrm{curl}\mathrm{E} = \mathrm{f}_1 \\ \varepsilon\partial_t \mathrm{E} - \mathrm{curl}\mathrm{H} = \mathrm{f}_2 \end{cases}$$

A space-time setting for linear hyperbolic operators

Basic assumptions

We consider a linear operator in space $\mathrm{A} \in \mathcal{L}(\mathcal{D}(\mathrm{A}), H)$ with domain $\mathcal{D}(\mathrm{A}) \subset H$ where:

$\Omega \subseteq \mathbb{R}^{\mathrm{D}}$ is a bounded Lipschitz domain

$H \subseteq \mathrm{L}_2(\Omega; \mathbb{R}^J)$ is a Hilbert space

$(\mathrm{v}, \mathrm{w})_H = (\mathrm{M}\mathrm{v}, \mathrm{w})_{0,\Omega}$ is weighted inner product

$\mathrm{M} \in \mathrm{L}_\infty(\Omega, \mathbb{R}^{J\times J}_{sym})$ is uniformly positive

Homogeneous boundary conditions

Elastic waves	$\mathcal{D}(\mathrm{A}) = H(\mathrm{div}, \Omega; \mathbb{R}^{\mathrm{D}\times\mathrm{D}}_{sym}) \times H^1_0(\Omega; \mathbb{R}^{\mathrm{D}})$
Acoustic waves	$\mathcal{D}(\mathrm{A}) = H^1(\Omega) \times H_0(\mathrm{div}, \Omega)$
Electro-magnetic waves	$\mathcal{D}(\mathrm{A}) = H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$
Skew-adjoint	$(\mathrm{Av}, \mathrm{w})_{0,\Omega} = -(\mathrm{v}, \mathrm{Aw})_{0,\Omega} \qquad \mathrm{v}, \mathrm{w} \in \mathcal{D}(\mathrm{A})$

The variational setting

In abstract setting we consider the operator

$$L = \mathrm{M}\partial_t + \mathrm{A}$$

on the space-time cylinder $Q = \Omega \times (0, T)$. We observe that

$$(L\mathrm{v}, \mathrm{w})_{0,Q} = -(\mathrm{v}, L\mathrm{w})_{0,Q}, \quad \mathrm{v}, \mathrm{w} \in \mathrm{C}_0^1(Q; \mathbb{R}^{\mathrm{J}}).$$

Depending on L we define the space

$$\mathrm{H}(L, Q) = \{\mathrm{v} \in \mathrm{L}_2(Q; \mathbb{R}^{\mathrm{J}}) : \mathrm{g} \in \mathrm{L}_2(Q; \mathbb{R}^{\mathrm{J}}) \text{ exists with} \\ (\mathrm{g}, \mathrm{w})_{0,Q} = -(\mathrm{v}, L\mathrm{w})_{0,Q} \text{ for all } \mathrm{w} \in \mathrm{C}_0^1(Q; \mathbb{R}^{\mathrm{J}})\}.$$

Now we extend L to this space and $\mathrm{H}(L, \Omega)$ is a Hilbert space w.r.t. norm

$$\|\mathrm{v}\|_{L,Q} = \sqrt{\|\mathrm{v}\|_{0,Q}^2 + \|L\mathrm{v}\|_{0,Q}^2}$$

To study operator L in $\mathcal{L}(\mathrm{V},\mathrm{W})$ and the evolution equation

$$L\mathrm{u} = \mathrm{f}$$

we fix the following setting

$\mathrm{V} \subset \mathrm{H}(L,Q)$ is the closure of $\{\mathrm{v} \in \mathrm{C}^1([0,T];\mathrm{D}(\mathrm{A})) : \mathrm{v}(0) = 0\}$

$\mathrm{W} = \overline{L(\mathrm{V})}$ is a subspace of $\mathrm{L}_2(Q;\mathbb{R}^{\mathrm{J}})$

$\|\mathrm{w}\|_{\mathrm{W}}^2 = (\mathrm{M}\mathrm{w},\mathrm{w})_{0,Q}$ is weighted norm

$\|\mathrm{v}\|_{\mathrm{V}}^2 = \|\mathrm{v}\|_{\mathrm{W}}^2 + \|\mathrm{M}^{-1}L\mathrm{v}\|_{\mathrm{W}}^2$ is weighted graph norm on V

This process extends to initial values $\mathrm{u}_0 \neq 0$ by replacing $\mathrm{f}(t)$ in

$$\mathrm{M}\partial_t\mathrm{u} + \mathrm{A}\mathrm{u} = \mathrm{f}$$

with $\mathrm{f}(t) - \mathrm{A}\mathrm{u}_0$.

The variational setting

Babuska-Aziz theorem

Let V and W be Hilbert spaces.Then a linear mapping $L : \mathrm{V} \longrightarrow \mathrm{W}'$ is an isomorphism if and only if the associated form $a : \mathrm{V} \times \mathrm{W} \longrightarrow \mathbb{R}$ satisfies the following conditions:

(i) Continuity there exists $\mathrm{C} \geq 0$ s.t

$$|a(\mathrm{v}, \mathrm{w})| \leq \mathrm{C} \|\mathrm{v}\|_{\mathrm{V}} \|\mathrm{w}\|_{\mathrm{W}} \tag{1}$$

(ii) Inf-sup condition there exists $\alpha > 0$ s.t

$$sup_{\mathrm{w} \in \mathrm{W}} \frac{a(\mathrm{v}, \mathrm{w})}{\|w\|_W} \geq \alpha \|\mathrm{v}\|_{\mathrm{V}} \qquad \text{for all} \mathrm{v} \in \mathrm{V} \tag{2}$$

(iii) For every $\mathrm{w} \in \mathrm{W}$, there exists $\mathrm{v} \in \mathrm{V}$ with $a(\mathrm{v}, \mathrm{w}) \neq 0$

The variational setting

Babuska-Aziz theorem

Supplement. If we assume only (i) and (ii) then

$$L : \mathrm{V} \longrightarrow \{\mathrm{w} \in \mathrm{W}; a(\mathrm{v}, \mathrm{w}) = 0 \text{ for all } \mathrm{v} \in \mathrm{V}\} \subset \mathrm{W}'$$

is an isomorphism. Moreover (2) is equivalent to

$$\|L\mathrm{v}\|_{\mathrm{W}'} \geq \alpha \|\mathrm{v}\|_{\mathrm{V}} \qquad \text{for all } \mathrm{v} \in \mathrm{V}$$

The name for condition (2) comes from the equivalent formulation

$$inf_{\mathrm{v} \in \mathrm{V}} sup_{\mathrm{w} \in \mathrm{W}} \frac{a(\mathrm{v}, \mathrm{w})}{\|v\|_{\mathrm{V}} \|w\|_{\mathrm{W}}} \geq \alpha > 0.$$

The variational setting

Lemma

Assume that $(\mathrm{Az},\mathrm{z})_{0,\Omega} \geq 0$ for $\mathrm{z} \in \mathrm{D(A)}$. Then, the bilinear form $\mathrm{b}(\cdot,\cdot)$ is continuous and inf-sup stable in $\mathrm{V} \times \mathrm{W}$ with $\beta = (4T^2+1)^{-1/2}$ i.e.,

$$sup_{\mathrm{w}\in\mathrm{W}\setminus\{0\}} \frac{b(\mathrm{v},\mathrm{w})}{\|\mathrm{w}\|_{\mathrm{W}}} \geq \beta\|\mathrm{v}\|_{\mathrm{V}} \quad \mathrm{v} \in \mathrm{V}$$

Theorem

For given $\mathrm{f} \in \mathrm{L}_2(Q;\mathbb{R}^{\mathrm{J}})$ there exists a unique solution $\mathrm{u} \in \mathrm{V}$ of

$$(L\mathrm{u},\mathrm{w})_{0,Q} = (\mathrm{f},\mathrm{w})_{0,Q} \quad \mathrm{w} \in \mathrm{W}$$

satisfying the a priori bound $\|\mathrm{u}\|_{\mathrm{V}} \leq \sqrt{4T^2+1}\|\mathrm{M}^{-1/2}\mathrm{f}\|_{0,Q}$

The variational setting

Remark

The approach presented here to show that $L \in \mathcal{L}(\mathrm{V}, \mathrm{W})$ is an invertible operator in suitable Hilbert space V and W only requires to show that L is injective.