A Space-Time Petrov-Galerkin method for Linear Wave Equations following [Wieners, 2016]
Talk No.3

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Overview

1. What do we want?
2. What to do?
3. Duality Based Error Estimation
   - The Primal and the Dual Problem
   - Localization of the Error in our Goal Functional
   - Nonlinear Goal Functionals
4. The Algorithm

What do we want?

1. Compute the solution u of the PDE

$$M\partial_t u + div(F(u)) = f$$

   with zero initial and boudary conditions.

2. Evaluate the functional E at u or other words compute $E(u)$

Goal Functional

We call this E goal functional.
Examples for goal functionals:

- $E_1(u) := \int_\Gamma u(T,x)dx$ for some $\Gamma \subseteq \Omega$,
- $E_2(u) := \int_{\tilde{Q}} Gu(t,x)d(t,x)$ for some $\tilde{Q} \subseteq Q$ and some linear operator G,
- $E_3(u) := E_1(u) + cE_2(u)$ for some constant $c \in \mathbb{R}$,
- $E_4(u) := E_1(u)^2 + E_2(u)^2 + E_3(u)^2$.

However in this presentation we just consider linear goal functionals as E_1, E_2 and E_3.
Since we consider E as linear we can write $\langle E, v\rangle := E(v)$.

What to do?

To compute $\langle E, u \rangle$ we first have to compute u.
However unfortunately this is usually not possible exactly.
Therefore we have to use an approximation u_h for u and we approximate $\langle E, u \rangle \approx \langle E, u_h \rangle$.

Wishes:

- high accuracy in our goal functional evaluation,
- low computational costs

Solution:

- adaptive mesh refinement for our goal functional.

However to refine adaptively we have to localize the error in the goal functional.

The Primal and The Dual Problem

First of all we consider

The Primal Problem

Find $u \in V$ such that

$$(Lu, w)_{0,Q} = (f, w)_{0,Q} \quad \forall w \in W,$$

and

The Dual Problem

Find $u^* \in V^*$ such that

$$(w, L^* u^*)_{0,Q} = \langle E, w \rangle \quad \forall w \in W.$$

Localization of the Error in our Goal Functional

Let u^* be sufficiently smooth and $w_h \in W_h$ then for our error holds

$$\langle E, u - u_h \rangle$$

$$= \sum_{R \in \mathcal{R}} (f - M\partial_t u_h - div(F(u_h)), u^*)_{0,R} + (n_K.(F(u_h) - F_K^{up}(u_h)), u^*)_{0,I \times \partial K}$$

$$= \sum_{R \in \mathcal{R}} (f - M\partial_t u_h - div(F(u_h)), u^* - w_h)_{0,R} +$$

$$(n_K.(F(u_h) - F_K^{up}(u_h)), u^* - w_h)_{0,I \times \partial K}$$

With Chauchy Schwarz inequality we obtain that

$$\langle E, u - u_h \rangle \leq \sum_{R \in \mathcal{R}} \|f - M\partial_t u_h - div(F(u_h))\|_{0,R} \|u^* - w_h\|_{0,R} +$$

$$\|n_K.(F(u_h) - F_K^{up}(u_h))\|_{0,I\times\partial K} \|u^* - w_h\|_{0,I\times\partial K} \quad \forall w_h \in W_h$$

So by choosing a projection $I_h : V^* \mapsto W_h$ we can localize our error estimator for the element $R \in \mathcal{R}$ which is defined by

$$\eta_R := \|f - M\partial_t u_h - div(F(u_h))\|_{0,R} \|u^* - I_h u^*\|_{0,R} +$$

$$\|n_K.(F(u_h) - F_K^{up}(u_h))\|_{0,I\times\partial K} \|u^* - I_h u^*\|_{0,I\times\partial K}.$$

However we do not know u^*!

But we can approximate u^* by using again a finite element method. We solve the Problem:

Find $u_h^* \in W_h$ such that

$$(Lv_h, u_h^*) = \langle E, v_h \rangle \quad \forall v_h \in V_h.$$

For the operator I_h we can choose a higher order recovery operator(or a lower order interpolation operator).

Nonlinear Goal Functionals

Now we know how to deal with linear goal functionals.
Let $E \in C^2(W)$ now be a nonlinear goal functional. Then we can represent the error as

$$E(u_h)-E(u) = \langle E'(u_h), u-u_h\rangle + \int_0^1 (1-s)E''(u_h+s(u-u_h))[u-u_h, u-u_h]ds$$

and hence the second term just depends on quadratic on $\|u-u_h\|_{0,Q}$ this can be neglected if E'' is bounded. Hence we can apply the Theory for linear goal functionals.

The Algorithm

1. choose low order polynomials on the initial mesh
2. **while** $\max_{R\in\mathcal{R}}(p_R) < p_{max}$ or $\max_{R\in\mathcal{R}}(\rho_R) < \rho_{max}$
3. compute u_h
4. compute u_h^* and $I_h u_h^*$
5. compute η_R for all $R \in \mathcal{R}$
6. if the error is small enough $(\sum_{R\in\mathcal{R}} \eta_R \leq TOL)$ STOP
7. mark cells $R \in \mathcal{R}$ for which holds that $\eta_R > \nu \max_{\tilde{R}\in\mathcal{R}}(\eta_{\tilde{R}})$
8. increase polynomial degrees on marked cells
9. **while end**

References

Christian Wieners (2016)
A space time Petrov-Galerkin Method for linear wave equations
Zurich Summer School 2016.

Thanks for your Attention