First Initial-Boundary Value Problem For The Heat Equation

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The Classical Problem

Let $f \in C(Q_T), f_i \in C^{1,0}(Q_T)$ and $\varphi \in C_0^1(\Omega)$.
Find $u(x,t) \in C^{2,1}(Q_T)$, with $Q_T := \Omega \times (0,T)$ for some bounded set Ω, such that:

$$\mathcal{M}_0 u := u_t - \Delta u = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \qquad (1)$$

with the initial condition

$$u(x,0) = \varphi(x) \quad \forall x \in \Omega, \qquad (2)$$

and the boundary condition

$$u(x,t) = 0 \quad \forall x \in S_T, \qquad (3)$$

with $S_T := S \times (0,T)$ where $S = \partial\Omega$, holds.

The Spaces $H^{\Delta,1}(Q_T)$ And $H_0^{\Delta,1}(Q_T)$

$H^{\Delta,1}(Q_T)$ denotes the space whose elements where $u, u_t, \nabla u$ and Δu are in $L^2(Q_T)$ or in other words if the norm

$$\|u\|^2_{H^{\Delta,1}(Q_T)} := \|u\|^2_{L^2(Q_T)} + \|u_t\|^2_{L^2(Q_T)} + \|\nabla u\|^2_{L^2(Q_T)} + \|\Delta u\|^2_{L^2(Q_T)},$$

is finite.
$H_0^{\Delta,1}(Q_T)$ denotes the elements in $H^{\Delta,1}(Q_T)$, where the trace with respect to space is 0. On this space we define the equivalent norm

$$\|u\|^2_{H_0^{\Delta,1}(Q_T)} := \|u_t\|^2_{L^2(Q_T)} + \|\Delta u\|^2_{L^2(Q_T)}.$$

The Problem For $H_0^{\Delta,1}(Q_T)$

Let $f \in L^2(Q_T), f_i \in H^{1,0}(Q_T)$ and $\varphi \in H_0^1(\Omega)$.
Find $u(x,t) \in H^{\Delta,1}(Q_T)$, with $Q_T := \Omega \times (0,T)$ for some bounded set Ω, such that:

$$\mathcal{M}_0 u := u_t - \Delta u = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}, \tag{4}$$

with the initial condition

$$u(x,0) = \varphi(x) \quad \forall x \in \Omega, \tag{5}$$

and the boundary condition

$$u(x,t) = 0 \quad \forall x \in S_T, \tag{6}$$

with $S_T := S \times (0,T)$ where $S = \partial\Omega$, holds in a weak sense.

The Problem As Operator Equation

Let $A : L^2(Q_T) \mapsto \mathcal{W}$ where $\mathcal{W} := L^2(Q_T) \times H_0^1(\Omega)$ where

$$Au := \{\mathcal{M}_0 u, u(\cdot, 0)\}.$$

Hence we can solve the problem by solving the operator equation

$$Au = \{F, \varphi\}$$

where $F := f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$.
On the Hilbert space $\mathcal{W}$ we define the inner product of $\{f_1, \psi_1\}$ and $\{f_2, \psi_2\}$ as

$$(\{f_1, \psi_1\}, \{f_2, \psi_2\})_{\mathcal{W}} := (f_1, f_2)_{L^2(Q_T)} + (\nabla\psi_1, \nabla\psi_2)_{L^2(Q_T)}$$

and the norm on $\mathcal{W}$ as $\|\cdot\|_{\mathcal{W}} := (\cdot, \cdot)_{\mathcal{W}}^{\frac{1}{2}}$.

The Operator A

As $\mathcal{D}(A)$ we take elements of the form

$$\psi(x) + \int_0^t \mathcal{X}(x,\tau)d\tau$$

where $\psi(x), \mathcal{X}(x,t) \in \mathcal{D}(\Delta)$ for almost all $t \in [0, T]$.
Here we can see that these elements are continuous in t.
As $\mathcal{D}(\Delta)$ we consider those elements $u \in H_0^1(\Omega)$ which solve

$$\Delta u = \tilde{f}$$

for some $\tilde{f} \in L^2(\Omega)$.

Now for an $v \in \mathcal{D}(A)$ it holds for $v = \psi(x) + \int_0^t \mathcal{X}(x,\tau)d\tau$

$$Av = \{\Delta\psi(x) + \mathcal{X}(x,t) + \int_0^t \Delta\mathcal{X}(x,\tau)d\tau, \psi\}.$$

Hence we can conclude that the operator A is densely defined on $L^2(Q_T)$. Furthermore we can show that for $v \in \mathcal{D}(A)$ it holds that

$$\|\nabla v(\cdot,t)\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(Q_t)} + \|\Delta v\|^2_{L^2(Q_t)} = \|\nabla v(\cdot,0)\|^2_{L^2(\Omega)} + \|\mathcal{M}_0 v\|^2_{L^2(Q_t)}.$$

The Closure Of A

Lemma

The operator A admits a closure $\overline{A}$.

This can be shown with the fact that an operator A admits a closure if and only if for any sequence v_m with $v_m \in \mathcal{D}(A)$ with $v_m \to 0$ and if $Av_m = \{f_m \psi_m\} \to \{f, \psi\}$ in the norm of $\mathcal{W}$ which can be written as

$$\|\{f, \psi\}\|_{\mathcal{W}}^2 = \|f\|_{L^2(Q_T)}^2 + \|\nabla\psi\|_{L^2(Q_T)}^2,$$

implies that $f = 0$ and $\psi = 0$.

From

$$\|\nabla v(\cdot,t)\|^2_{L^2(\Omega)} + \underbrace{\|v_t\|^2_{L^2(Q_t)} + \|\Delta v\|^2_{L^2(Q_t)}}_{=\|v\|_{H_0^{\Delta,1}(Q_T)}} = \underbrace{\|\nabla v(\cdot,0)\|^2_{L^2(\Omega)} + \|\mathcal{M}_0 v\|^2_{L^2(Q_t)}}_{=\|Av\|_{\mathcal{W}}}.$$

we can deduce that if Av_m (with $v_m \in \mathcal{D}(A)$) converges in $\mathcal{W}$ then also v_m converges in the norm of $H_0^{\Delta,1}(Q_T)$ and in the norm $\sup_{0\le t\le T} \|\nabla \cdot\|_{L^2(\Omega)}$. This shows that the elements in $\mathcal{D}(\overline{A})$ are contained in $H_0^{\Delta,1}(Q_T)$ and they are continuous in t in the $H_0^1(\Omega)$-norm. The operator $\overline{A}$ is defined on its domain as

$$\overline{A}v = \{\mathcal{M}_0 v, v(x,0)\}.$$

Furthermore it follows that $\overline{\mathcal{R}(A)} = \mathcal{R}(\overline{A})$.

Existence and Uniqueness of the Heat Equation

If we can additionally show that $\mathcal{R}(\overline{A}) = \mathcal{W}$ then $\overline{A}^{-1}$ exist and is bounded and the following theorem is valid:

Theorem

Let $F \in L^2(Q_T)$ and $\varphi \in H_0^1(\Omega)$ then the heat equation has a unique solution in $H_0^{\Delta,1}(Q_T)$. Furthermore the solution $u(x,t)$ is continous in t.

Generalized Solution in $L^2(Q_T)$

We understand $u \in L^2(Q_T)$ as generalized solution in $L^2(Q_T)$ if it fulfils

$$\int_{Q_T} u(\eta_t + \Delta\eta)dxdt + \int_\Omega \varphi\eta(x,0)dx = \int_{Q_T} -f\eta + \sum_{i=1}^{n} f_i \eta x_i dxdt$$

for all $\eta \in H_0^{\Delta,1}(Q_T)$ where $\eta(x, T) = 0$.

Theorem

The generalized solution in $L^2(Q_T)$ is unique if it exists.

Energy Balance and Integral Identity for Solutions in $H_0^{\Delta,1}(Q_T)$

From the equation

$$\int_{Q_t} \mathcal{M}_0 u.\eta dxdt = \int_{Q_t} \eta(f + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i})dxdt,$$

we obtain

$$\int_{Q_t} -u\eta_t + \nabla u.\nabla \eta dxdt - \int_{\Omega} \varphi\eta(x,0)dx + \int_{\Omega} u(x,t)\eta(x,t) =$$

$$= \int_{Q_t} f\eta - \sum_{i=1}^{n} f_i \eta x_i dxdt.$$

From

$$\int_{Q_t} \mathcal{M}_0 u.udxdt = \int_{Q_t} u(f + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i})dxdt,$$

we can deduce the Energy Balance

$$\frac{1}{2}\|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(Q_t)} = \int_{Q_t} fu - \boldsymbol{f}.\nabla udxdt + \frac{1}{2}\|u(\cdot,0)\|^2_{L^2(\Omega)}.$$

Here $\boldsymbol{f} := (f_1, f_2, \cdots, f_n)$.

We understand $u \in V_0^{1,0}(Q_T)$ as generalized solution in $V_0^{1,0}(Q_T)$ if it fulfils

$$\int_{Q_t} u\eta_t + \nabla u.\nabla \eta dxdt - \int_{\Omega} \varphi \eta(x,0)dx + \int_{\Omega} u(x,t)\eta(x,t) =$$

$$= \int_{Q_t} f\eta - \sum_{i=1}^{n} f_i \eta x_i dxdt$$

for all $\eta \in H_0^1(Q_T)$.
$V_0^{1,0}(Q_T) := C([0,T], L^2(\Omega)) \cap H_0^{1,0}(Q_T)$ is a Banach space provided with the norm $|u|_{Q_T} := \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}$.

Existence And Uniqueness In $V_0^{1,0}(Q_T)$

Theorem

Let $\varphi \in L^2(\Omega), f \in L^{2,1}(Q_T)$ and $f_i \in L^2(Q_T)$. Then there exists a unique generalized solution in $V_0^{1,0}(Q_T)$.

To show this theorem we take sequences $\varphi_m, f_m, f_{i,m}$ with $\varphi_m \in H_0^1(\Omega), f_m \in L^2(Q_T), f_{i,m} \in H^{1,0}(Q_T)$ such that $\varphi_m \to \varphi$ in $L^2(\Omega)$, $f_m \to f$ in $L^{2,1}(Q_T)$ and $f_{i,m} \to f_i$ in $L^2(Q_T)$. Then we show that the limit of this solution, which is unique, converges to an element in $V_0^{1,0}(Q_T)$. The uniqueness follows form the uniqueness of the generalized solution in $L^2(Q_T)$.

Thanks for your attention