Introduction to Maxwell's Equations

Exercise 4

Problem 1
Let H be a Hilbert space and let $\mathrm{M} : D(\mathrm{M}) \subset \mathsf{H} \to \mathsf{H}$ be linear and selfadjoint with reduced operator

$$\mathcal{M} : D(\mathcal{M}) := D(\mathrm{M}) \cap \overline{R(\mathrm{M})} \subset \overline{R(\mathrm{M})} \to \overline{R(\mathrm{M})}.$$

(This implies that M and $\mathcal{M}$ are lddc. Why?) Moreover, let $D(\mathcal{M}) \hookrightarrow\!\!\!\rightarrow \mathsf{H}$ be compact and let $\lambda \in \mathbb{R} \setminus \{0\}$. Show:

(i) $0 \notin \sigma_{\mathsf{p}}(\mathrm{M}) \Rightarrow 0 \in \rho(\mathrm{M})$.

(ii) $0 \in \rho(\mathcal{M})$.

(iii) $N(\mathrm{M} - \lambda) = N(\mathcal{M} - \lambda)$ is finite dimensional.

(iv) $\sigma(\mathrm{M}) \setminus \{0\} = \sigma_{\mathsf{p}}(\mathrm{M}) \setminus \{0\} = \sigma_{\mathsf{p}}(\mathcal{M}) = \sigma(\mathcal{M})$ is discrete, i.e., $\sigma(\mathcal{M})$ has no accumulation point in $\mathbb{R}$.

Hint: (iii) and (iv) can be proved by the same indirect argument.

Problem 2
Let H_1, H_2, H_3 be Hilbert spaces and let

$$\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2, \quad \mathrm{A}_2 : D(\mathrm{A}_2) \subset \mathsf{H}_2 \to \mathsf{H}_3$$

be lddc operators with <u>closed ranges</u> and adjoints

$$\mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1, \quad \mathrm{A}_2^* : D(\mathrm{A}_2^*) \subset \mathsf{H}_3 \to \mathsf{H}_2\,.$$

The corresponding reduced operators are

$$\mathcal{A}_1 : D(\mathcal{A}_1) := D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset \overline{R(\mathrm{A}_1^*)} \to \overline{R(\mathrm{A}_1)}, \quad \mathcal{A}_2 : D(\mathcal{A}_2) := D(\mathrm{A}_2) \cap \overline{R(\mathrm{A}_2^*)} \subset \overline{R(\mathrm{A}_2^*)} \to \overline{R(\mathrm{A}_2)}$$

with adjoints

$$\mathcal{A}_1^* : D(\mathcal{A}_1^*) := D(\mathrm{A}_1^*) \cap \overline{R(\mathrm{A}_1)} \subset \overline{R(\mathrm{A}_1)} \to \overline{R(\mathrm{A}_1^*)}, \quad \mathcal{A}_2^* : D(\mathcal{A}_2^*) := D(\mathrm{A}_2^*) \cap \overline{R(\mathrm{A}_2)} \subset \overline{R(\mathrm{A}_2)} \to \overline{R(\mathrm{A}_2^*)}.$$

All operators are lddc and $(\mathrm{A}_1, \mathrm{A}_1^*)$, $(\mathrm{A}_2, \mathrm{A}_2^*)$ as well as $(\mathcal{A}_1, \mathcal{A}_1^*)$, $(\mathcal{A}_2, \mathcal{A}_2^*)$ define dual pairs. We define $D_2 := D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*)$. Moreover, let the sequence or complex property be satisfied, that is

$$\mathrm{A}_2\,\mathrm{A}_1 \subset 0 \quad \text{, i.e.,} \quad R(\mathrm{A}_1) \subset N(\mathrm{A}_2).$$

Note that then also $\mathrm{A}_1^*\,\mathrm{A}_2^* \subset 0$, i.e., $R(\mathrm{A}_2^*) \subset N(\mathrm{A}_1^*)$ holds. In other words, we have the following sequences or complexes:

$$D(\mathrm{A}_1) \xrightarrow{\ \mathrm{A}_1\ } D(\mathrm{A}_2) \xrightarrow{\ \mathrm{A}_2\ } \mathsf{H}_3$$
$$\mathsf{H}_1 \xleftarrow{\ \mathrm{A}_1^*\ } D(\mathrm{A}_1^*) \xleftarrow{\ \mathrm{A}_2^*\ } D(\mathrm{A}_2^*)$$

Let us consider the following saddle point problem. For given (f, g) find (x, y), such that

$$\begin{aligned} \mathrm{A}_2^*\,\mathrm{A}_2\,y + \mathrm{A}_1\,x + \pi_2 y &= g, \\ \mathrm{A}_1^*\,y &= f. \end{aligned} \tag{1}$$

Show:

(i) There exists a unique pair

$$(x, y) \in D(\mathcal{A}_1) \times \big(D_2 \cap D(\mathrm{A}_2^* \mathrm{A}_2)\big) \subset R(\mathrm{A}_1^*) \times \mathsf{H}_2$$

solving (1), if and only if $(f, g) \in R(\mathrm{A}_1^*) \times \mathsf{H}_2$. The solution (x, y) depends continuously on the data (f, g).

(ii) $R(\mathrm{A}_1)$ is closed, if and only if the inf/sup-conditions

$$\begin{aligned} 0 < \frac{1}{c_1} &= \inf_{0 \neq x \in D(\mathcal{A}_1)} \sup_{y \in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1 x, y\rangle_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1} |y|_{\mathsf{H}_2}} \\ &= \inf_{0 \neq x \in D(\mathcal{A}_1)} \sup_{y \in D(\mathrm{A}_1^*)} \frac{\langle \mathrm{A}_1 x, y\rangle_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1} |y|_{\mathsf{H}_2}} = \inf_{0 \neq x \in D(\mathcal{A}_1)} \sup_{y \in D(\mathcal{A}_1^*)} \frac{\langle \mathrm{A}_1 x, y\rangle_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1} |y|_{\mathsf{H}_2}} \end{aligned}$$

hold.

(iii) $R(\mathrm{A}_2)$ is closed, if and only if the inf/sup-conditions

$$0 < \frac{1}{c_2} = \inf_{0 \neq y \in D(\mathcal{A}_2)} \sup_{z \in D(\mathrm{A}_2^*)} \frac{\langle \mathrm{A}_2 y, z\rangle_{\mathsf{H}_3}}{|y|_{\mathsf{H}_2} |z|_{\mathsf{H}_3}} = \inf_{0 \neq y \in D(\mathcal{A}_2)} \sup_{z \in D(\mathcal{A}_2^*)} \frac{\langle \mathrm{A}_2 y, z\rangle_{\mathsf{H}_3}}{|y|_{\mathsf{H}_2} |z|_{\mathsf{H}_3}}$$

hold.

Remark: The bilinear form $(\varphi, \psi) \mapsto a(\varphi, \psi) := \langle \mathrm{A}_2 \,\cdot\,, \mathrm{A}_2 \,\cdot\,\rangle_{\mathsf{H}_3}$ is coercive over

$$D(\mathcal{A}_2) = D(\mathrm{A}_2) \cap R(\mathrm{A}_2^*) = D(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp}$$

by the Poincaré type estimate, if and only if $R(\mathrm{A}_2)$ is closed, and ${\mathcal{A}_1}^{-1}$ and $(\mathcal{A}_1^*)^{-1}$ exist as continuous operators, if and only if $R(\mathrm{A}_1)$ is closed. Hence, by the latter results the solution theory holds, if and only if the following two conditions are satisfied:

(1) a is coercive over $D(\mathrm{A}_2) \cap N(\mathrm{A}_1^*) \cap K_2^{\perp}$.

(2) The inf/sup-condition

$$0 < \frac{1}{c_1} = \inf_{0 \neq x \in D(\mathcal{A}_1)} \sup_{y \in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1 x, y\rangle_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1} |y|_{\mathsf{H}_2}}$$

holds.

These conditions are implied by the following two classical Babuska/Brezzi conditions:

(1') a is coercive over $D(\mathrm{A}_2) \cap N(\mathrm{A}_1^*)$.

(2') The inf/sup-condition

$$0 < \frac{1}{c_1} = \inf_{0 \neq x \in D(\mathcal{A}_1)} \sup_{y \in D(\mathrm{A}_2)} \frac{\langle \mathrm{A}_1 x, y\rangle_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1} |y|_{\mathsf{H}_2}}$$

holds.

(1'), (2') are stronger assumptions than (1), (2) as they do not handle the cohomology group K_2. For $K_2 = \{0\}$ they are equivalent.