Introduction to Maxwell's Equations

Exercise 3

Problem 1

Let $\Omega \subset \mathbb{R}^3$ be open. Show that the extensions by zero $\hat{u}$, $\hat{E}$, resp. $\hat{H}$ of some $u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega)$, $E \in \overset{\circ}{\mathsf{R}}(\Omega)$, resp. $H \in \overset{\circ}{\mathsf{D}}(\Omega)$ belong to $\mathsf{H}^1(\mathbb{R}^3)$, $\mathsf{R}(\mathbb{R}^3)$, resp. $\mathsf{D}(\mathbb{R}^3)$.

Problem 2

Let $B, D \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $f \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3, \mathbb{R})$. Show the following Poincaré (potential) formulas:

(i) The function $u \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3)$ defined by

$$u(x) := \int_0^1 x \cdot B(tx) \,\mathrm{d}\, t$$

satisfies

$$\nabla\, u(x) = B(x) + \int_0^1 tx \times (\operatorname{rot} B)(tx) \,\mathrm{d}\, t.$$

Especially, for B with $\operatorname{rot} B = 0$ it holds $\nabla\, u = B$.

Note the following: Let $\varphi \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R})$ be defined by $\varphi(t) := \tilde{u}(tx)$ with some $\tilde{u} \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3, \mathbb{R})$. Then $\varphi'(t) = x \cdot (\tilde{\nabla} u)(tx)$ and hence for $\tilde{B} := \nabla\, \tilde{u} \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3, \mathbb{R}^3)$, which implies $\operatorname{rot} \tilde{B} = 0$,

$$\tilde{u}(x) - \tilde{u}(0) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) \,\mathrm{d}\, t = \int_0^1 x \cdot \tilde{B}(tx) \,\mathrm{d}\, t.$$

(ii) The vector field $E \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3)$ defined by

$$E(x) := -\int_0^1 tx \times D(tx) \,\mathrm{d}\, t$$

satisfies

$$\operatorname{rot} E(x) = D(x) - \int_0^1 t^2 x (\operatorname{div} D)(tx) \,\mathrm{d}\, t.$$

Especially, for D with $\operatorname{div} D = 0$ it holds $\operatorname{rot} E = D$.

(ii) The vector field $H \in \overset{\circ}{\mathsf{C}}{}^\infty(\mathbb{R}^3)$ defined by

$$H(x) := \int_0^1 t^2 x f(tx) \,\mathrm{d}\, t$$

satisfies $\operatorname{div} H = f$.

Note: There might be some wrong signs in the formulas, which have to be corrected as well. ;)
Hint for (ii): First, prove the formula

$$\begin{aligned}\operatorname{rot}(B \times D) &= \partial_D B - (\operatorname{div} B) D + (\operatorname{div} D) B - \partial_B D \\ &= \sum_{n=1}^{3} D_n\, \partial_n\, B - (\operatorname{div} B) D + (\operatorname{div} D) B - \sum_{n=1}^{3} B_n\, \partial_n\, D.\end{aligned}$$

Problem 3
Let H_1 and H_2 be Hilbert spaces and let

$$\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

be a lddc operator. Show that

$$\mathrm{M} := \begin{bmatrix} 0 & \mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix} : D(\mathrm{M}) := D(\mathrm{A}) \times D(\mathrm{A}^*) \subset \mathsf{H}_1 \times \mathsf{H}_2 \to \mathsf{H}_1 \times \mathsf{H}_2; \quad (x, y) \mapsto (\mathrm{A}^* y, \mathrm{A}\, x)$$

is selfadjoint. Similarly, show that

$$\mathrm{M}_- := \begin{bmatrix} 0 & -\mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}, \qquad i\,\mathrm{M}_- := i \begin{bmatrix} 0 & -\mathrm{A}^* \\ \mathrm{A} & 0 \end{bmatrix}$$

is skew-selfadjoint resp. selfadjoint.