Introduction to Maxwell's Equations

Exercise 2

Let H_1, H_2, H_3 be Hilbert spaces and let

$$\mathrm{A}_1 : D(\mathrm{A}_1) \subset \mathsf{H}_1 \to \mathsf{H}_2, \quad \mathrm{A}_2 : D(\mathrm{A}_2) \subset \mathsf{H}_2 \to \mathsf{H}_3$$

be lddc operators with adjoints

$$\mathrm{A}_1^* : D(\mathrm{A}_1^*) \subset \mathsf{H}_2 \to \mathsf{H}_1, \quad \mathrm{A}_2^* : D(\mathrm{A}_2^*) \subset \mathsf{H}_3 \to \mathsf{H}_2,$$

which are then also lddc. The corresponding reduced operators are

$$\mathcal{A}_1 : D(\mathcal{A}_1) := D(\mathrm{A}_1) \cap \overline{R(\mathrm{A}_1^*)} \subset \overline{R(\mathrm{A}_1^*)} \to \overline{R(\mathrm{A}_1)}, \quad \mathcal{A}_2 : D(\mathcal{A}_2) := D(\mathrm{A}_2) \cap \overline{R(\mathrm{A}_2^*)} \subset \overline{R(\mathrm{A}_2^*)} \to \overline{R(\mathrm{A}_2)}$$

with adjoints

$$\mathcal{A}_1^* : D(\mathcal{A}_1^*) := D(\mathrm{A}_1^*) \cap \overline{R(\mathrm{A}_1)} \subset \overline{R(\mathrm{A}_1)} \to \overline{R(\mathrm{A}_1^*)}, \quad \mathcal{A}_2^* : D(\mathcal{A}_2^*) := D(\mathrm{A}_2^*) \cap \overline{R(\mathrm{A}_2)} \subset \overline{R(\mathrm{A}_2)} \to \overline{R(\mathrm{A}_2^*)}.$$

All operators are lddc and $(\mathrm{A}_1, \mathrm{A}_1^*)$, $(\mathrm{A}_2, \mathrm{A}_2^*)$ as well as $(\mathcal{A}_1, \mathcal{A}_1^*)$, $(\mathcal{A}_2, \mathcal{A}_2^*)$ define dual pairs. Moreover, let the sequence or complex property be satisfied, that is

$$\mathrm{A}_2 \, \mathrm{A}_1 = 0 \quad \text{, i.e.,} \quad R(\mathrm{A}_1) \subset N(\mathrm{A}_2).$$

Note that then also $\mathrm{A}_1^* \, \mathrm{A}_2^* = 0$, i.e., $R(\mathrm{A}_2^*) \subset N(\mathrm{A}_1^*)$ holds. In other words, we have the following sequences or complexes:

$$D(\mathrm{A}_1) \xrightarrow{\mathrm{A}_1} D(\mathrm{A}_2) \xrightarrow{\mathrm{A}_2} \mathsf{H}_3$$
$$\mathsf{H}_1 \xleftarrow{\mathrm{A}_1^*} D(\mathrm{A}_1^*) \xleftarrow{\mathrm{A}_2^*} D(\mathrm{A}_2^*)$$

By the projection theorem it holds

$$\begin{aligned} \mathsf{H}_1 &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} \overline{R(\mathrm{A}_1^*)}, \\ \mathsf{H}_2 &= N(\mathrm{A}_2) \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A}_2^*)} = \overline{R(\mathrm{A}_1)} \oplus_{\mathsf{H}_2} N(\mathrm{A}_1^*), \\ \mathsf{H}_3 &= \overline{R(\mathrm{A}_2)} \oplus_{\mathsf{H}_3} N(\mathrm{A}_2^*). \end{aligned}$$

Problem 1

Show

$$\begin{aligned} D(\mathrm{A}_1) &= N(\mathrm{A}_1) \oplus_{\mathsf{H}_1} D(\mathcal{A}_1), \\ D(\mathrm{A}_2) &= N(\mathrm{A}_2) \oplus_{\mathsf{H}_2} D(\mathcal{A}_2), \qquad & D(\mathrm{A}_1^*) &= D(\mathcal{A}_1^*) \oplus_{\mathsf{H}_2} N(\mathrm{A}_1^*), \\ D(\mathrm{A}_2^*) &= D(\mathcal{A}_2^*) \oplus_{\mathsf{H}_3} N(\mathrm{A}_2^*) \end{aligned}$$

and hence $R(\mathrm{A}_1) = R(\mathcal{A}_1)$, $R(\mathrm{A}_2) = R(\mathcal{A}_2)$, $R(\mathrm{A}_1^*) = R(\mathcal{A}_1^*)$, $R(\mathrm{A}_2^*) = R(\mathcal{A}_2^*)$.

Problem 2
Show the refined Helmholtz type decompositions

$$\begin{aligned}
\mathsf{H}_2 &= \overline{R(\mathrm{A}_1)} \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A}_2^*)}, & K_2 &:= N(\mathrm{A}_2) \cap N(\mathrm{A}_1^*),\\
N(\mathrm{A}_2) &= \overline{R(\mathrm{A}_1)} \oplus_{\mathsf{H}_2} K_2, & N(\mathrm{A}_1^*) &= K_2 \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A}_2^*)},\\
\overline{R(\mathcal{A}_1)} = \overline{R(\mathrm{A}_1)} &= N(\mathrm{A}_2) \ominus_{\mathsf{H}_2} K_2, & \overline{R(\mathcal{A}_2^*)} = \overline{R(\mathrm{A}_2^*)} &= N(\mathrm{A}_1^*) \ominus_{\mathsf{H}_2} K_2,\\
D(\mathrm{A}_2) &= \overline{R(\mathrm{A}_1)} \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} D(\mathcal{A}_2), & D(\mathrm{A}_1^*) &= D(\mathcal{A}_1^*) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} \overline{R(\mathrm{A}_2^*)},\\
D_2 &= D(\mathcal{A}_1^*) \oplus_{\mathsf{H}_2} K_2 \oplus_{\mathsf{H}_2} D(\mathcal{A}_2), & D_2 &:= D(\mathrm{A}_2) \cap D(\mathrm{A}_1^*).
\end{aligned}$$

Problem 3
The embeddings $D(\mathcal{A}_1) \hookrightarrow \mathsf{H}_1$, $D(\mathcal{A}_2) \hookrightarrow \mathsf{H}_2$, and $K_2 \hookrightarrow \mathsf{H}_2$ are compact, if and only if the embedding $D_2 \hookrightarrow \mathsf{H}_2$ is compact. In this case, K_2 has finite dimension and all ranges and complexes are closed.

Problem 4
Let $R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ be closed and let K_2 be finite dimensional, which is satisfied, if, e.g., $D_2 \hookrightarrow \mathsf{H}_2$ is compact. Show that the linear first order system

$$\begin{aligned}
\mathrm{A}_2\, x &= f,\\
\mathrm{A}_1^*\, x &= g,\\
\pi_2\, x &= k.
\end{aligned} \tag{1}$$

is uniquely solvable in D_2, if and only if $f \in R(\mathrm{A}_2)$, $g \in R(\mathrm{A}_1^*)$, and $k \in K_2$. Moreover, the unique solution $x \in D_2$ depends continuously on the data, i.e., $|x|_{\mathsf{H}_2} \leq c_{\mathrm{A}_2}|f|_{\mathsf{H}_3} + c_{\mathrm{A}_1}|g|_{\mathsf{H}_1} + |k|_{\mathsf{H}_2}$.

Problem 5
Find variational formulations to compute the (partial) solutions of (1).

Problem 6
Let $R(\mathrm{A}_1)$ and $R(\mathrm{A}_2)$ be closed and let K_2 be finite dimensional, which is satisfied, if, e.g., $D_2 \hookrightarrow \mathsf{H}_2$ is compact. Show that the linear second order system

$$\begin{aligned}
\mathrm{A}_2^*\, \mathrm{A}_2\, x &= f,\\
\mathrm{A}_1^*\, x &= g,\\
\pi_2\, x &= k.
\end{aligned} \tag{2}$$

is uniquely solvable in

$$\tilde{D}_2 := D(\mathrm{A}_2^*\, \mathrm{A}_2) \cap D(\mathrm{A}_1^*) = \big\{ \xi \in D_2 \,:\, \mathrm{A}_2\, \xi \in D(\mathrm{A}_2^*) \big\},$$

if and only if $f \in R(\mathrm{A}_2^*)$, $g \in R(\mathrm{A}_1^*)$, and $k \in K_2$. Moreover, the unique solution $x \in \tilde{D}_2$ depends continuously on the data, i.e., $|x|_{\mathsf{H}_2} \leq c_{\mathrm{A}_2}^2|f|_{\mathsf{H}_2} + c_{\mathrm{A}_1}|g|_{\mathsf{H}_1} + |k|_{\mathsf{H}_2}$.

Problem 7
Find variational formulations to compute the (partial) solutions of (2).