Introduction to Maxwell's Equations

Exercise 1

Let H_1, H_2 be Hilbert spaces and let

$$\mathrm{A} : D(\mathrm{A}) \subset \mathsf{H}_1 \to \mathsf{H}_2$$

be a linear (possibly unbounded), densely defined and closed (*lddc*) operator with adjoint

$$\mathrm{A}^* : D(\mathrm{A}^*) \subset \mathsf{H}_2 \to \mathsf{H}_1,$$

which is then also lddc. It holds $\mathrm{A}^{**} = \mathrm{A}$, i.e., $(\mathrm{A}, \mathrm{A}^*)$ is a dual pair. Moreover, let

$$\mathcal{A} : D(\mathcal{A}) := D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \subset \overline{R(\mathrm{A}^*)} \to \overline{R(\mathrm{A})}, \quad \mathcal{A}^* : D(\mathcal{A}^*) := D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})} \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}$$

be the reduced operators, which are also lddc and define a dual pair as well.

Problem 1
Show that indeed for

$$\mathcal{B} : D(\mathcal{B}) := D(\mathrm{A}^*) \cap \overline{R(\mathrm{A})} \subset \overline{R(\mathrm{A})} \to \overline{R(\mathrm{A}^*)}, \quad y \mapsto \mathrm{A}^* y$$

it holds $\mathcal{B} = \mathcal{A}^*$ and hence $(\mathcal{A}, \mathcal{A}^*)$ is a dual pair.

Problem 2
Show that the following assertions are equivalent:

(i) $\exists\, c_{\mathrm{A}} \in (0, \infty) \quad \forall\, x \in D(\mathcal{A}) \qquad |x|_{\mathsf{H}_1} \leq c_{\mathrm{A}} |\,\mathrm{A}\, x|_{\mathsf{H}_2}$

(i*) $\exists\, c_{\mathrm{A}^*} \in (0, \infty) \quad \forall\, y \in D(\mathcal{A}^*) \qquad |y|_{\mathsf{H}_2} \leq c_{\mathrm{A}^*} |\,\mathrm{A}^*\, y|_{\mathsf{H}_1}$

(ii) $R(\mathrm{A}) = R(\mathcal{A})$ is closed in H_2.

(ii*) $R(\mathrm{A}^*) = R(\mathcal{A}^*)$ is closed in H_1.

(iii) $(\mathcal{A})^{-1} : R(\mathrm{A}) \to D(\mathcal{A})$ is continuous and bijective with norm bounded by $(1 + c_{\mathrm{A}}^2)^{1/2}$.

(iii*) $(\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to D(\mathcal{A}^*)$ is continuous and bijective with norm bounded by $(1 + c_{\mathrm{A}^*}{}^2)^{1/2}$.

From now on, let us always choose the "best" Friedrichs/Poincaré type constants in Problem 2, i.e., $c_{\mathrm{A}}, c_{\mathrm{A}^*} \in (0, \infty]$ are given by the Rayleigh quotients

$$\frac{1}{c_{\mathrm{A}}} := \inf_{0 \neq x \in D(\mathcal{A})} \frac{|\,\mathrm{A}\, x|_{\mathsf{H}_2}}{|x|_{\mathsf{H}_1}}, \qquad \frac{1}{c_{\mathrm{A}^*}} := \inf_{0 \neq y \in D(\mathcal{A}^*)} \frac{|\,\mathrm{A}^*\, y|_{\mathsf{H}_1}}{|y|_{\mathsf{H}_2}}.$$

Problem 3
Show that $c_{\mathrm{A}} = c_{\mathrm{A}^*}$.

Problem 4
Show

$$\big|(\mathcal{A})^{-1}\big|_{R(\mathrm{A}), R(\mathrm{A}^*)} = c_{\mathrm{A}} = \big|(\mathcal{A}^*)^{-1}\big|_{R(\mathrm{A}^*), R(\mathrm{A})}$$

and

$$\big|(\mathcal{A})^{-1}\big|^2_{R(\mathrm{A}), D(\mathcal{A})} = 1 + c_{\mathrm{A}}^2 = \big|(\mathcal{A}^*)^{-1}\big|^2_{R(\mathrm{A}^*), D(\mathcal{A}^*)}.$$

Problem 5
Let $D(\mathcal{A}) = D(\mathrm{A}) \cap \overline{R(\mathrm{A}^*)} \hookrightarrow \mathsf{H}_1$ be compact. Show that then the assertions of Problem 2 hold. Especially, the ranges $R(\mathrm{A})$ and $R(\mathrm{A}^*)$ are closed. Moreover, show that the inverse operators

$$\mathcal{A}^{-1} : R(\mathrm{A}) \to R(\mathrm{A}^*), \qquad (\mathcal{A}^*)^{-1} : R(\mathrm{A}^*) \to R(\mathrm{A})$$

are compact.

Problem 6
Show: The embedding $D(\mathcal{A}) \hookrightarrow \mathsf{H}_1$ is compact, if and only if the embedding $D(\mathcal{A}^*) \hookrightarrow \mathsf{H}_2$ is compact.