WEB-SPLINES

concept by K. Höllig, U. Reif, J. Wipper

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For the course - Spezialvorlesung: Isogeometric Analysis

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Classification and chronology

- attempt to unify CAD/CAM and FEM
- meshless finite element technique
- within the standard Ritz-Galerkin framework
- introduced in 2001 by Höllig, Reif and Wipper (compare IgA by Hughes in 2005)

Outline

Definition

- Weighted splines
- Extended splines
- WEB-splines

Approximation properties and stability

Numerical examples

Further remarks

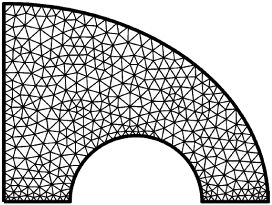
DEFINITION

DEFINITION

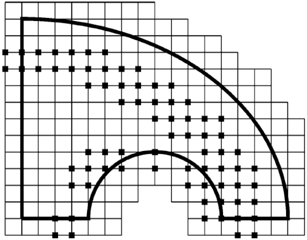
WEIGHTED SPLINES

Meshless method

FEM-mesh

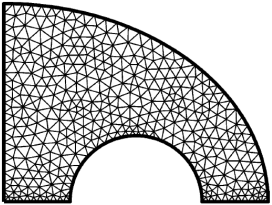


tensor-product
basis functions



Meshless method

FEM-mesh



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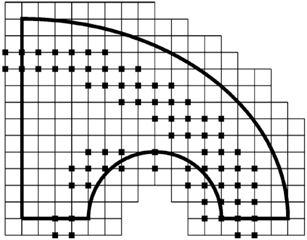


How to fulfill Dirichlet boundary conditions?

Weight functions

- use tensor product B-splines b_k
- multiply basis functions b_k by a smoothed version ω of the distance function to $\Gamma_D \subset \partial D$
- obtain subspace

$$\text{span}\{\omega b_k : D \cap \text{supp}(b_k) \neq \emptyset\}$$

  which confirms to the BC
- IDEA: Kantorowitsch and Krylow, 1956

How to choose appropriate weight functions?

assume boundary is defined in terms of simple algebraic equations

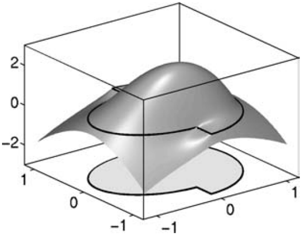
use R-Function Method (RFM) by Rvachev et al. (1995, 2000)

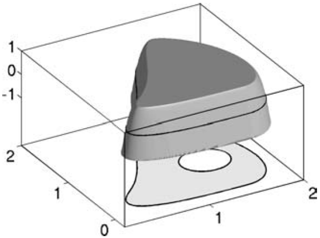
Set operation	Corresponding R-function
Complement: D^c	$r_c(\omega) = -\omega$
Intersection: $D_1 \cap D_2$	$r_\cap(\omega_1, \omega_2) = \omega_1 + \omega_2 - \sqrt{\omega_1^2 + \omega_2^2}$
Union: $D_1 \cup D_2$	$r_\cup(\omega_1, \omega_2) = \omega_1 + \omega_2 + \sqrt{\omega_1^2 + \omega_2^2}$

$\rightarrow$ Example on the blackboard

Weight functions

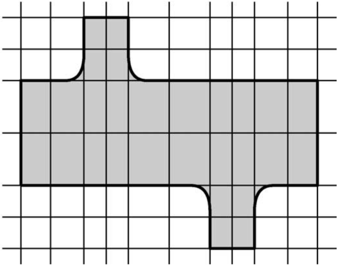
- other weight functions might be possible
- R-functions are still complicated
- one does not need to compute the explicite form
- for this application some kind of plateau would be useful

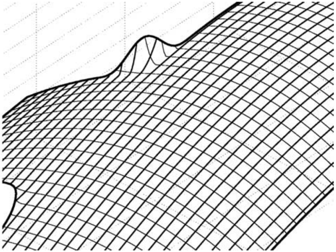




Weighted B-splines

Problem with stability:
Due to possible small areas near the boundary, coefficients in these areas might become large $\rightarrow$ system ill conditioned





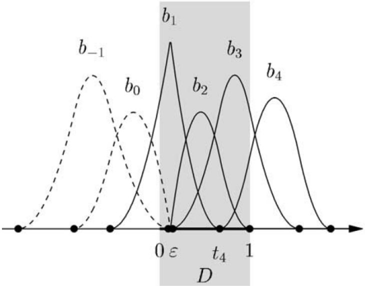
DEFINITION

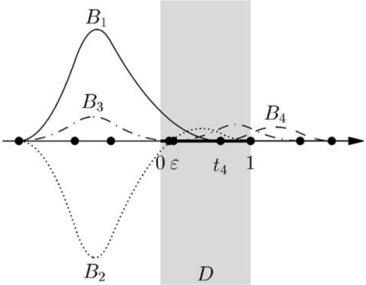
EXTENDED SPLINES

Notation

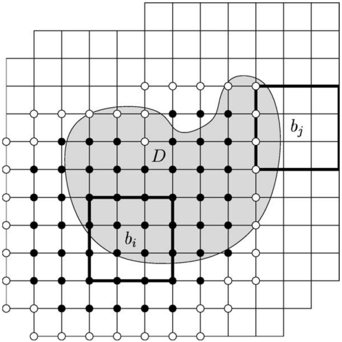
b_k	...	B-splines
λ_k	...	dual functionals for b_k
B_k	...	extended B-splines
Q_k	...	grid cells in support of b_k
$K = \{k : \Omega \cap \text{supp}(b_k) \neq \emptyset\}$	...	set of indices of basis fcts.
$I = \{i : b_i \text{ contains an inner grid cell}\}$	...	set of inner indices
$J = K \setminus I$	...	set of outer indices
$I(j) = \{i \in I : Q_j \subset \text{supp}(b_i)\}$	...	related inner indices
$J(i) = \{j \in J : i \in I(j)\}$	...	related outer indices
$p_{i,j}$	...	polynomial which agrees with b_i on Q_j

Stability problem - Example in 1D





Extended B-splines - 2D basis supports



Relevant biquadratic B-splines for a domain D

Extended B-splines (eb-splines)

overcome stability problem by suitably joining outer B-splines to inner B-splines

Definition

For $i \in I$, $j \in J(i)$ we define the extension coefficients

$$e_{i,j} := \lambda_j p_{i,j}.$$

Then the extended B-splines are

$$B_i := b_i + \sum_{j \in J(i)} e_{i,j} b_j, \quad i \in I.$$

How do we get the extension coefficients?

- uniform grid: via Lagrange interpolation

$$e_{i,j} = \prod_{\substack{\nu=0 \\ \ell+\nu\neq i}}^{n} \frac{j-\ell-\nu}{i-\ell-\nu}$$

- nonuniform grid: more difficult
  - ☐ generate $p_{i,j}$ in Taylor form
  - ☐ expand at an arbitrary point $\tau_j \in Q_j$
  - ☐ apply dual funcional (de Boor and Fix)

$$\lambda_k f = \sum_{\ell=0}^{n} (-1)^{n-\ell} \psi_k^{(n-\ell)}(\tau_k) f^{(\ell)}(\tau_k), \quad \psi_k^{(n-\ell)}(x) = \frac{1}{n!} \prod_{\ell=1}^{n} (t_{k+\ell} - x)$$

Extended B-splines (eb-splines)

One can show, that eb-splines inherit the properties

- locality
- boundedness
- existence of dual functionals
- polynomial precision

from standard B-splines which are crucial for approximation purposes.

DEFINITION

WEB-SPLINES

Combine wb-splines and eb-splines

Definition

For $i \in I$, $j \in J(i)$ let the extension coefficients $e_{i,j}$ be as before. Further let ω be a positive weight function, smooth on D and equivalent to some power of the boundary distance function

$$\omega(x) \asymp \mathsf{dist}(x, \partial D)^r.$$

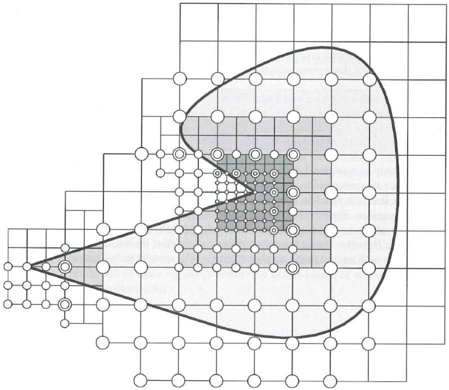
Then the extended B-splines are

$$B_i := \frac{\omega}{\omega(x_i)} \left(b_i + \sum_{j \in J(i)} e_{i,j} b_j \right), \quad i \in I,$$

where x_i is the center of the interior grid cell Q_i.

Different types

uniform grid, nonuniform grid, hierarchical bases (Kraft), ...



APPROXIMATION PROPERTIES AND STABILITY

Some useful remarks

- extension coefficients are uniformly bounded

$$|e_{i,j}| \preceq 1$$

- support of web-splines is larger BUT only a small strip of inner cells is effected, consequently

$$\text{supp}(B_i) \preceq h$$

- web-splines are linearly independent and have the dual functional

$$\Lambda_k = \frac{\omega(x_k)}{\omega}\lambda_k, \quad k \in I$$

Stability

web-splines and the dual functional are uniformly bounded and biorthogonal

$$||B_i||_0 \preceq 1 \quad ||\Lambda_k||_0 \preceq 1 \quad \int_D B_i \Lambda_k = \delta_{i,k}$$

Stability

For a weight function of order γ linear combinations of web-splines satisfy

$$\Big|\Big|\sum_{i\in I} c_i B_i\Big|\Big|_0 \asymp h^{m/2}||C||,$$

where the constants depend on D, ω and n.

Quasi-interpolant

standard projector

$$P_h u = \sum_{i \in I} \left(\int \lambda_i u \right) B_i$$

if ω is an ℓ-regular weight function of order γ

$$||P_h u||_{\nu, Q \cap D} \leq \text{const}(D, \omega, n) h^{-\nu} ||u||_{0,Q'}, \quad \nu \leq \min(\ell, n)$$

where Q' is the union of the supports of all web-splines which are nonzero on $Q \cap D$.

Approximation order

if ω is an ℓ-regular weight function of order γ and $v = u/w$ is smooth on $\bar{D}$, then

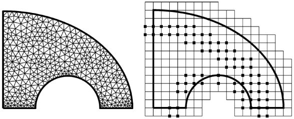
$$||u - P_h u||_\nu \leq \text{const}(D, \omega, u, n) h^{n+1-\nu}$$

for $\nu \leq \min(\ell, n)$

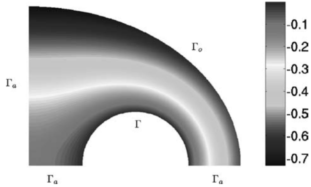
$\rightarrow$ optimal approximation order.

NUMERICAL EXAMPLES

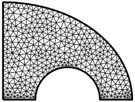
Heat conduction problem

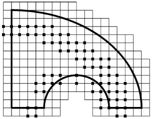


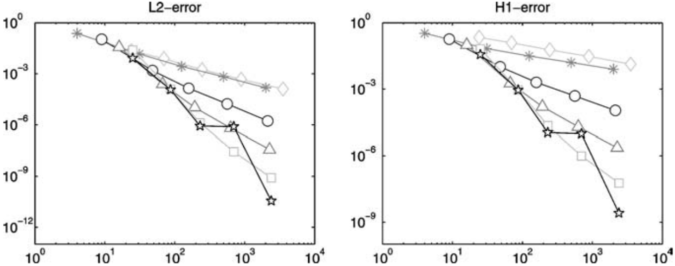
$$
\begin{aligned}
\Delta u &= 0 && \text{in } D \\
u &= u_0 && \text{on } \Gamma \\
\partial_\perp u &= -\gamma(u - u_1) && \text{on } \Gamma_0 \\
\partial_\perp u &= 0 && \text{on } \Gamma_a
\end{aligned}
$$



Heat conduction problem







Relative errors for hat-functions (⋄) and Web-spline approximations of degree $n = 1, 2, 3, 4, 5$ (markers $*, \circ, \triangle, \square, \star$) as a function of the dimension d.

Further examples in the papers

- linear elasticity
- thin plates
- scattered data approximation

FURTHER REMARKS

Multigrid methods

for uniform B-spline basis:

grid transfer of web-splines (lecture: prolongation matrix)

$$\tilde{B}_\ell \ldots \text{ grid width } 2h \qquad B_\ell \ldots \text{ grid width } h$$

$$P_h \tilde{B}_\ell = \sum_{i \in I} p_{i,\ell} B_i, \quad p_{i,\ell} = \frac{\omega(x_i)}{\omega(\tilde{x}_\ell)} \left(s_{i-2\ell} + \sum_{j \in \tilde{J}(\ell)} \tilde{e}_{\ell,j} s_{i-2j} \right)$$

where s are the coefficients from multivariate B-spline subdivision

Application in IgA

competitor to IgA?

Application in IgA

competitor to IgA?

Yes: alternative approach, closer to FEM, mathematical insights stability/ approximation properties similarly proven as for FEM

No: combination possible (e.g. trimmed surfaces, surfaces with holes ...), chance to use advantages of both methods

Advantages and Difficulties

Advantages stated by the authors

- no mesh generation
- homogeneous BC matched exactly
- accurate solutions with low-dimensional subspaces
- smoothness and approximation order can be chosen arbitrarily
- hierarchical basis permit adaptive refinement
- parallelization and multigrid possible

Advantages and Difficulties

Difficulties

- tradeoff between difficulties
  - FEM: meshing
  - web-splines: weight functions
  - IgA: parameterization of domain
- deciding whether grid cell is completely inside the domain
- most of the advantages are valid for IgA as well

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