Analysis-suitable adaptive T-mesh refinement with linear complexity (by [Morgenstern & Peterseim, 2015])

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Analysis-suitable t-splines of arbitrary degree: Definition, linear independence and approximation properties.

Axioms of adaptivity.

Mathematical analysis of variational isogeometric methods.

[4] Li, X., and Scott, M. A.
Analysis-suitable t-splines: Characterization, refineability, and approximation.

Analysis-suitable adaptive t-mesh refinement with linear complexity.
Overview

1. Adaptive mesh-refinement
2. Analysis-suitability
3. The overlay
4. Nestedness
5. Linear complexity
Overview

T-splines can be used for local refinement, but in general the T-splines are not linearly independent. To have this property, we need *analysis-suitable* T-meshes.

The proposed refinement algorithm provides the following:

1. the preservation of analysis-suitability and nestedness of the generated T-spline spaces,
2. a bounded cardinality of the overlay,
3. linear computational complexity of the refinement procedure.
Adaptive mesh refinement

We consider only a 2D-index domain, as the physical domain can be obtained via a suitable mapping.

**Definition 1 (Initial mesh, element).**
Given positive numbers $M, N \in \mathbb{N}$, the initial mesh $G_0$ is a tensor product mesh consisting of closed squares (also denoted elements) with side length 1, i.e.

\[
G_0 := \left\{ [m-1,m] \times [n-1,n] : m \in \{1, \ldots, M\}, n \in \{1, \ldots, N\} \right\}.
\]

**Definition 2.**
The level of an element $K$ is defined by

\[
\ell(K) := - \log_2 |K|
\]
(p,q)-patches

**Definition 3.**

Given an element $K$ and polynomial degrees $p$ and $q$, the $(p,q)$-patch is defined by

$$\mathcal{G}^{p,q}(K) := \{K \in \mathcal{G} : \text{Dist}(K', K) \leq D^{p,q}(l(K))\}$$

with

$$D^{p,q}(k) := \begin{cases} 2^{-k/2}([p/2] + \frac{1}{2}, \lceil q/2 \rceil + \frac{1}{2}) & \text{if } k \text{ is even} \\ 2^{-(k+1)/2}([p/2] + \frac{1}{2}, 2\lceil q/2 \rceil + 1) & \text{if } k \text{ is odd} \end{cases}$$

Note that $\text{Dist}(K', K)$ is the vector-valued distance between the midpoints of $K'$ and $K$. 
Example for a \((p,q)\)-patch
Refining an element

From now on, we assume $p, q \geq 2$. This ensures that neighbouring elements of $K$ are always in $G^{p,q}(K)$ and nested elements $K \subseteq \hat{K}$ have nested $(p,q)$-patches, i.e. $G^{p,q}(K) \subseteq G^{p,q}(\hat{K})$.

**Definition 4.**

Given an arbitrary element $K = [\mu, \mu + \tilde{\mu}] \times [\nu, \nu + \tilde{\nu}]$, we define the operators

$$\text{bisect}_x(K) := [\mu, \mu + \frac{\tilde{\mu}}{2}] \times [\nu, \nu + \tilde{\nu}], [\mu + \frac{\tilde{\mu}}{2}, \mu + \tilde{\mu}] \times [\nu, \nu + \tilde{\nu}]$$

and

$$\text{bisect}_y(K) := [\mu, \mu + \tilde{\mu}] \times [\nu, \nu + \frac{\tilde{\nu}}{2}], [\mu, \mu + \tilde{\mu}] \times [\nu + \frac{\tilde{\nu}}{2}, \nu + \tilde{\nu}]$$.
Definition 5 (Bisection, Multiple bisections).

Given a mesh $\mathcal{G}$ and an element $K \in \mathcal{G}$, we denote by $\text{bisect}(\mathcal{G}, K)$ the mesh that results from a level dependent bisection of $K$,

$$\text{bisect}(\mathcal{G}, K) := \mathcal{G} \setminus K \cup \text{child}(K),$$

with

$$\text{child}(K) := \begin{cases} 
\text{bisect}_x(K), & \text{if } \ell(K) \text{ is even} \\
\text{bisect}_y(K), & \text{if } \ell(K) \text{ is odd}
\end{cases}.$$

The bisection of multiple elements $\mathcal{M} \subseteq \mathcal{G}$ is defined by successive bisections in an arbitrary order, i.e.

$$\text{bisect}(\mathcal{G}, \mathcal{M}) := \text{bisect}(\text{bisect}(\ldots \text{bisect}(\mathcal{G}, K_1)), K_J).$$
Now we can define our refinement algorithm but first need some superset of $\mathcal{M}$.

**Algorithm (Closure).**

Given a mesh $\mathcal{G}$ and a set of marked elements $\mathcal{M} \subseteq \mathcal{G}$ to be bisected, the closure $\text{clos}^{p,q}_{\mathcal{G}}(\mathcal{M})$ of $\mathcal{M}$ is computed as follows

$$\tilde{\mathcal{M}} := \mathcal{M}$$

repeat

for all $K \in \tilde{\mathcal{M}}$ do

$$\tilde{\mathcal{M}} := \tilde{\mathcal{M}} \cup \{K' \in \mathcal{G}^{p,q}(K) : l(K') < l(K)\}$$

end for

until $\tilde{\mathcal{M}}$ stops growing

return $\text{clos}^{p,q}_{\mathcal{G}}(\mathcal{M}) = \tilde{\mathcal{M}}$
The refinement algorithm

**Algorithm** (*Refinement*).

Given a mesh $\mathcal{G}$ and a set of marked elements $\mathcal{M} \subseteq \mathcal{G}$ to be bisected, $\text{ref}^{p,q}(\mathcal{G}, \mathcal{M})$ is defined by

$$\text{ref}^{p,q}(\mathcal{G}, \mathcal{M}) := \text{biset}(\mathcal{G}, \text{clos}^{p,q}_G(\mathcal{M})).$$

In the following examples, the polynomial degrees are $p = q = 3$ and $M = 4, N = 5$. 
Examples
Examples
Definition 6 \(((p, q)\text{-admissible bisections})\).
Given a mesh \(G\) and an element \(K \in G\), the bisection of \(K\) is called \((p,q)\text{-admissible}\) if all \(K' \in G^{p,q}\) satisfy \(l(K') \geq l(K)\).
In the case of several elements \(M = \{K_1, \ldots, K_J\} \subseteq G\), the bisection \(\text{bisect}(G, M)\) is \((p,q)\text{-admissible}\) if there is an order \((\sigma(1), \ldots, \sigma(J))\) (this is, if there is a permutation \(\sigma\) of \(1, \ldots, J\)) such that

\[
\text{bisect}(G, M) = \text{bisect}(... \text{bisect}(G, K_{\sigma(1)}), ...) , K_{\sigma(J)})
\]
is a concatenation of \((p,q)\text{-admissible bisections})\.
Definition 7 (Admissible refinement).

A refinement $G$ of $G_0$ is $(p,q)$-admissible if there is a sequence of meshes $G_1, \ldots, G_J = G$ and markings $M_j \subseteq G_j$ for $j = 0, \ldots, J - 1$, such that $G_{j+1} = \text{bisection}(G_j, M_j)$ is a $(p,q)$-admissible bisection for all $j = 0, \ldots, J - 1$. The set of all $(p,q)$-admissible meshes, which is the initial mesh and all its admissible refinements, is denoted by $\mathbb{A}^{p,q}$. 
Preservance of admissibility

Lemma 8 (Local quasi-uniformity).

Given $K \in \mathcal{G} \in A^{p,q}$, any $K' \in \mathcal{G}^{p,q}(K)$ satisfies $\ell(K') \geq \ell(K) - 1$.

Proof.
See [5].

Proposition 9.

Any admissible mesh $\mathcal{G}$ and any set of marked elements $\mathcal{M} \subseteq \mathcal{G}$ satisfy $\text{ref}^{p,q}(\mathcal{G}, \mathcal{M}) \in A^{p,q}$.

Proof.
By induction and with Lemma 8. For details, see [5].
Analysis-suitability

To ensure that the T-spline blending functions of a refined mesh are still linearly independent, we need the concept of analysis-suitable meshes.

**Definition 10.**

Consider an admissible mesh \( G \in A^{p,q} \). The set of vertices of \( G \) is denoted by \( N \). We define the active region

\[
AR := \left[ \left\lceil \frac{p}{2} \right\rceil, M - \left\lceil \frac{p}{2} \right\rceil \right] \times \left[ \left\lceil \frac{q}{2} \right\rceil, N - \left\lceil \frac{q}{2} \right\rceil \right]
\]

and the set of active nodes \( N_A := N \cap AR \).

**Definition 11.**

We denote by \( hSk \) (resp. \( vSk \)) the horizontal (resp. vertical) skeleton, which is the union of all horizontal (resp. vertical) edges. Note that \( hSk \cap vSk = N \).

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A.S. adaptive T-mesh refinement

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**Definition 12 (T-junction extension).**

Denote by $\mathcal{T} \subset \mathcal{N}_A$ the set of all active nodes with valence three and refer to them as T-junctions. Consider a T-junction $T = (t_1, t_2) \in \mathcal{T}$ of type $\dashv$. Clearly, $t_1$ is one of the entries of $X(t_2)$. Then extract from $X(t_2)$ the $p+1$ consecutive indices $i_{-\lceil p/2 \rceil}, \ldots, i_{\lceil p/2 \rceil}$ such that $i_0 = t_1$. We denote

$$
\text{ext}^p_q(T) := [i_{-\lceil p/2 \rceil}, i_0] \times \{t_2\}, \quad \text{ext}^p_f(T) := (i_0, i_{\lceil p/2 \rceil}] \times \{t_2\},
$$

$$
\text{ext}^p_q(T) := \text{ext}^p_f(T) \cup \text{ext}^p_q(T).
$$

Here $X(y) := \{z \in [0, M] : (z, y) \in vSk\}$ is a global index set (analogous definition for the $x$-direction).

**Definition 13 (Analysis-suitality).**

A mesh is analysis-suitable if horizontal T-junction extensions do not intersect vertical T-junction extensions.
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$$
\text{ext}_{e}^{p,q}(T) := [i_{\lfloor p/2 \rfloor}, i_0] \times \{t_2\},
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A mesh is analysis-suitable if horizontal T-junction extensions do not intersect vertical T-junction extensions.
**Theorem 14.**

*All admissible meshes are analysis suitable.*

**Proof.**

By induction over admissible bisections. For details, see [5].

**Corollary 15.**

*All admissible meshes provide T-spline blending functions that are non-negative, linearly independent, and form a partition of unity. Moreover, on each element* $K \in \mathcal{G} \in \Delta^{p,q}$, *there are not more than* $2(p + 1)(q + 1)$ *T-spline basis functions that have a support on* $K$.

**Proof.**

See [3, 1]
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**Proof.**

See [3, 1]
To use this algorithm for a posteriori error-driven refinement, we need some theoretical properties on the overlay, which is the common coarsest refinement.

**Definition 16 (Overlay).**
We define the operator $\text{Min}_{\subseteq}$ which yields all minimal elements of a set that is partially ordered by "$\subseteq$",

$$\text{Min}_{\subseteq}(\mathcal{M}) := \{K \in \mathcal{M} : \forall K' \in \mathcal{M} : K' \subseteq K \rightarrow K' = K\}$$

The overlay of $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ is defined by

$$\mathcal{G}_1 \otimes \mathcal{G}_2 := \text{Min}_{\subseteq}(\mathcal{G}_1 \cup \mathcal{G}_2)$$
Proposition 17.

$G_1 \otimes G_2$ is the coarsest refinement of $G_1$ and $G_2$ in the sense that for any $\hat{G}$ being a refinement of $G_1$ and $G_2$, and $G_1 \otimes G_2$ being a refinement of $\hat{G}$, it follows that $\hat{G} = G_1 \otimes G_2$.

Proof.

Blackboard, see [5].
Proposition 17.

$G_1 \otimes G_2$ is the coarsest refinement of $G_1$ and $G_2$ in the sense that for any $\hat{G}$ being a refinement of $G_1$ and $G_2$, and $G_1 \otimes G_2$ being a refinement of $\hat{G}$, it follows that $\hat{G} = G_1 \otimes G_2$.

Proof.

Blackboard, see [5].
Two theoretical properties

**Proposition 18.**
For any admissible meshes $G_1, G_2 \in \mathbb{A}^{p,q}$, the overlay $G_1 \otimes G_2$ is also admissible.

**Lemma 19.**
For all $G_1, G_2 \in \mathbb{A}^{p,q}$ holds

$$\#(G_1 \otimes G_2) + \#G_0 \leq \#G_1 + \#G_2.$$  

The second property is an assumption in [2].
Now the nesting behaviour of the T-spline spaces corresponding to admissible meshes is investigated (for details, see [5, 4]).

**Definition 20.**

For any partitions $G_1, G_2$ of $\bar{\Omega}$ we introduce the refinement relation “$\leq$”, which is defined using the overlay

$$G_1 \leq G_2 \iff G_1 \otimes G_2 = G_2$$

**Corollary 21.**

Denote the skeleton of a mesh $G$ by $Sk(G) := hSk(G) \cup vSk(G)$. Then for rectangular partitions $G_1, G_2$ of $\bar{\Omega}$ holds the equivalence

$$G_1 \leq G_2 \iff Sk(G_1) \subseteq Sk(G_2)$$
Nestedness

Now the nesting behaviour of the T-spline spaces corresponding to admissible meshes is investigated (for details, see [5, 4]).

**Definition 20.**

For any partitions $G_1, G_2$ of $\overline{\Omega}$ we introduce the refinement relation $\leq$, which is defined using the overlay

$$G_1 \preceq G_2 \iff G_1 \otimes G_2 = G_2$$

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Denote the skeleton of a mesh $G$ by $Sk(G) := hSk(G) \cup vSk(G)$. Then for rectangular partitions $G_1, G_2$ of $\overline{\Omega}$ holds the equivalence

$$G_1 \preceq G_2 \iff Sk(G_1) \subseteq Sk(G_2)$$
Definition 22.

Given a rectangular partition $\mathcal{G}$ of $\overline{\Omega}$, denote by $\text{ext}^{p,q}(\mathcal{G})$ the union of all $T$-junction extensions in the mesh $\mathcal{G}$. Then the extended mesh $\mathcal{G}^{\text{ext}}$ is defined as the unique rectangular partition of $\overline{\Omega}$ such that

$$Sk(\mathcal{G}^{\text{ext}}) = Sk(\mathcal{G}) \cup \text{ext}^{p,q}(\mathcal{G}).$$

Sketch of an extended mesh.
**Definition 23.**

Given a partition $G$ of $\overline{\Omega}$ into axis-aligned rectangles, we define by $Ptb(G)$ the set of all continuous and invertible mappings $\delta : \overline{\Omega} \rightarrow \overline{\Omega}$ such that the corners $(0,0), (M,0), (M,N), (0,N)$ are fixed points of $\delta$ and

$$\delta(G) = \{ \delta(K) : K \in G \}$$

is also a partition of $\overline{\Omega}$ into axis-aligned rectangles.

Note for $\delta \in Ptb(G)$, the corresponding skeleton satisfies $Sk(\delta(G)) = \delta(Sk(G))$. In general, such a perturbation $\delta$ does not map T-junction extensions to the corresponding extensions in the perturbed mesh, i.e.

$$ext_{\delta(G)}^{p,q}(\delta(T)) \neq \delta(ext_{G}^{p,q}(T)).$$
**Mesh perturbation**

**Definition 23.**

Given a partition $G$ of $\overline{\Omega}$ into axis-aligned rectangles, we define by $Ptb(G)$ the set of all continuous and invertible mappings $\delta : \overline{\Omega} \to \overline{\Omega}$ such that the corners $(0, 0), (M, 0), (M, N), (0, N)$ are fixed points of $\delta$ and

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$$\text{ext}_{\delta(G)}^{p,q}(\delta(T)) \neq \delta(\text{ext}_{G}^{p,q}(T)).$$
Example for a perturbation.
**Theorem 24.**

Given two analysis-suitable meshes $G_1$ and $G_2$, if for all $\delta \in Ptb(G_2)$ holds

$$(\delta(G_1))^{\text{ext}} \preceq (\delta(G_2))^{\text{ext}}$$

then the T-spline spaces corresponding to $G_1$ and $G_2$ are nested.

**Proof.**

See [4].
Theorem 24.

Given two analysis-suitable meshes $G_1$ and $G_2$, if for all $\delta \in \text{Pt}(G_2)$ holds

$$ (\delta(G_1))^{\text{ext}} \preceq (\delta(G_2))^{\text{ext}} $$

then the T-spline spaces corresponding to $G_1$ and $G_2$ are nested.

Proof.

See [4].
Theorem 25.

Any two meshes $G_1, G_2 \in \mathbb{A}^{p,q}$ that are nested in the sense $G_1 \preceq G_2$ satisfy for all $\delta \in \text{Ptb}(G_2)$

$$(\delta(G_1))^{\text{ext}} \preceq (\delta(G_2))^{\text{ext}}.$$ 

Proof.

It is sufficient to show

$$\text{ext}^{p,q}(\delta(G_1)) \cup \text{Sk}(\delta(G_1)) \subseteq \text{ext}^{p,q}(\delta(G_2)) \cup \text{Sk}(\delta(G_2)).$$

First, let $K \in G_1 \in \mathbb{A}^{p,q}$ and $G_2 := \text{biset}(G_1)$, then $G_1 \preceq G_2 \Rightarrow \text{Sk}(\delta(G_1)) \subseteq \text{Sk}(\delta(G_2))$.

The second part includes comparison of different cases. For further details, see [5].
**Theorem 25.**

Any two meshes $G_1, G_2 \in \mathbb{A}^{p,q}$ that are nested in the sense $G_1 \preceq G_2$ satisfy for all $\delta \in Ptb(G_2)$

$$\left(\delta(G_1)\right)^{ext} \preceq \left(\delta(G_2)\right)^{ext}.$$ 

**Proof.**

It is sufficient to show

$$\text{ext}^{p,q}(\delta(G_1)) \cup Sk(\delta(G_1)) \subseteq \text{ext}^{p,q}(\delta(G_2)) \cup Sk(\delta(G_2)).$$

First, let $K \in G_1 \in \mathbb{A}^{p,q}$ and $G_2 := \text{bise}t(G_1)$, then

$G_1 \preceq G_2 \Rightarrow Sk(\delta(G_1)) \subseteq Sk(\delta(G_2)).$

The second part includes comparison of different cases. For further details, see [5].
Combination of these two results gives us:

**Corollary 26.**

*For any two meshes \( G_1, G_2 \in \mathbb{A}^{p,q} \) that are nested in the sense \( G_1 \preceq G_2 \), the corresponding T-spline spaces are also nested.*
Linear complexity

The following estimate shows that the number of refined elements depends at most linearly on the number of marked elements.

**Theorem 27.**

*Any sequence of admissible meshes $G_0, G_1, \ldots, G_J$ with*

$$G_j = \text{ref}^{p,q}(G_{j-1}, M_{j-1}), \quad M_{j-1} \subseteq G_{j-1} \text{ for } j \in \{1, \ldots, J\}$$

*satisfies*

$$|G_J \setminus G_0| \leq C_{p,q} \sum_{j=0}^{J-1} |M_j|$$

*with $C_{p,q} = (3 + \sqrt{2})(4d_p + 1)(4d_q + \sqrt{2})$ and*

$$d_p = (1 + 2^{-1/2})p + 1 + \frac{5}{4}\sqrt{2}, \quad d_q = (1 + \sqrt{2})q + \frac{3}{2} + \sqrt{2}.$$
Sketch of proof

One can show the following:

- for $K \in \bigcup A^{p,q}$ and $\tilde{K} \in \mathcal{M}$, define $\lambda(K, \tilde{K})$ by
  $$\lambda(K, \tilde{K}) := \begin{cases} 2^{\ell(K) - \ell(\tilde{K})/2}, & \text{if } \ell(K) \leq \ell(\tilde{K}) + 1 \text{ and } \text{Dist}(K, \tilde{K}) \leq 2^{1 - \ell(K)/2}(d_p, d_q) \\ 0 & \text{otherwise} \end{cases}$$

- for all $j \in \{0, \ldots, J - 1\}$ and $\tilde{K} \in \mathcal{M}_j$ holds
  $$\sum_{K \in \mathcal{G}_J \setminus \mathcal{G}_0} \lambda(K, \tilde{K}) \leq (3 + \sqrt{2})(4d_p + 1)(4d_q + \sqrt{2}) = C_{p,q},$$

- each $K \in \mathcal{G}_J \setminus \mathcal{G}_0$ satisfies
  $$\sum_{\tilde{K} \in \mathcal{M}} \lambda(K, \tilde{K}) \geq 1.$$
Remarks and Examples

- The result of the theorem is not trivial, as there is no uniform bound for the number of generated elements.
- The large constant $C_{p,q}$ was not observed in the numerical experiments by the authors.

Generated and marked elements for randomly refined (3,3)-admissible meshes.
### Observed bounds for higher degrees of \((p,q)\)

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Maximal observed ratios for random refinement.
### Observed bounds for higher degrees of \((p,q)\)

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Maximal observed ratios when refining the lower left.
Thank you!