Guaranteed and sharp a posteriori error estimates in isogeometric analysis (following the paper [Kleiss &Tomar 2015])

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Overview

References

Sergey Repin (2008)
A Posteriori Estimates for Partial Differential Equations
Radon Series on Computational and Applied Mathematics 4.
Walter de Gruyter GmbH & Co. KG, 10785 Berlin, Germany.

Kleiss, Stefan K. and Tomar, Satyendra K.
Guaranteed and Sharp a Posteriori Error Estimates in Isogeometric Analysis
Computers & Mathematics with Applications, Volume 70, Issue 3, August 2015, Pages 167190
http://dx.doi.org/10.1016/j.camwa.2015.04.011

The second reference is denoted by [Kleiss &Tomar 2015].

The Model Problem

For $\Omega \subset \mathbb{R}^2$ the Model Problem is given by:
Find $u \in V_g$ such that

$$a(u,v) = \langle F, v \rangle \quad \forall v \in V_0,$$

where $V_0 := H_0^1(\Omega)$ and $V_g := g + V_0$. Here

$$a(u,v) := \int_\Omega (A(x).\nabla u(x)).\nabla v(x) \mathrm{d}\boldsymbol{x},$$

and

$$\langle F, v \rangle := \int_\Omega f(x) v(x) \mathrm{d}\boldsymbol{x},$$

where $A(x)$ is positive definite, bounded, symmetric and has a bounded inverse $A^{-1}(x)$ for all $x \in \Omega$ and $f \in L^2(\Omega)$.

Therefore we can define the norms

$$\|u\|_A := \sqrt{\int_\Omega (A(x).u(x)).u(x)\mathrm{d}\boldsymbol{x}},$$

and

$$\|u\|_{A^{-1}} := \sqrt{\int_\Omega (A^{-1}(x).u(x)).u(x)\mathrm{d}\boldsymbol{x}},$$

for a vector valued function u, which are equivalent to the norm in $L^2(\Omega)$. It obviously holds

$$\|u\|_A = \|Au\|_{A^{-1}}.$$

An Error Estimate

Theorem

Let $u \in V_g$ be the exact solution of the model problem and let $u_h \in V_h$ be an approximate solution. Furthermore let C_Ω be the constant from the "Friedrichs' like inequality" $\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_A$ for all $v \in V_0$. Then it holds

$$\|\nabla u - \nabla u_h\|_A \leq \|A\nabla u_h - y\|_{A^{-1}} + C_\Omega \|f + div(y)\|_{L^2(\Omega)} \quad \forall y \in H(div, \Omega).$$

For the proof see

Sergey Repin (2008)
A Posteriori Estimates for Partial Differential Equations
Radon Series on Computational and Applied Mathematics 4.
Walter de Gruyter GmbH & Co. KG, 10785 Berlin, Germany.

Hint for Proof: Take a look at the problem $a(w, v) = \langle f + div(y), v\rangle$.

Local Error Estimator 1

If we choose y as $A\nabla u_h$ it follows immediately that

$$\|\nabla u - \nabla u_h\|_A \leq C_\Omega \|f + div(y)\|_{L^2(\Omega)},$$

and therefore we can choose our local error estimators for a cell Q as

$$\eta_Q := \|div(y) + f\|_{L^2(Q)}.$$

Now can use some marking strategy which marks all cell Q which fulfill that

$$\eta_Q > \Theta,$$

where Θ is some bound as for example choosen in such a way that at least 20% are marked. However a numerical example in [Kleiss &Tomar 2015] showed that this error estimator overestimates the error and even has a lower convergence rate as the exact error.

Global Minimization Strategy

From

$$\|\nabla u - \nabla u_h\|_A \leq \|A\nabla u_h - y\|_{A^{-1}} + C_\Omega \|f + div(y)\|_{L^2(\Omega)} \quad \forall y \in H(div, \Omega),$$

it follows that

$$\|\nabla u - \nabla u_h\|_A^2 \leq \underbrace{\underbrace{(1+\beta)}_{:=a_1} \underbrace{\|A\nabla u_h - y\|_{A^{-1}}^2}_{:=B_1} + \underbrace{(1+\frac{1}{\beta})C_\Omega^2}_{:=a_2} \underbrace{\|f + div(y)\|_{L^2(\Omega)}^2}_{:=B_2}}_{:=M_\oplus^2(\beta,y)},$$

holds for all $\beta > 0$ and $y \in H(div, \Omega)$. Obviously our majorant $M_\oplus(\beta, y)$ fulfills

$$M_\oplus^2(\beta, y) = a_1 B_1 + a_2 B_2.$$

But how sharp is this estimate?

Definition

We say a sequence of finite dimensional subspaces $\{Y_j\}_{j=1}^{\infty}$ *of a Banachspace* Y *is* **limit dense** *if for all* $\varepsilon > 0$ *holds that there exists an index* j_ε *such that for all* $k \geq j_\varepsilon$ *and for all* $y \in Y$ *there exists a* $p_k \in Y_k$ *such that*

$$\|y - p_k\|_Y < \varepsilon.$$

Theorem

Let $\{Y_j\}_{j=1}^{\infty}$ *be limit dense in* $H(div, \Omega)$ *then*

$$\lim_{j \to \infty} \inf_{y \in Y_j, \beta > 0} M_\oplus^2(\beta, y) = \|\nabla u - \nabla u_h\|_A^2.$$

One can even show that $a_1 B_1 \to \|\nabla u - \nabla u_h\|_A^2$ and $a_2 B_2 \to 0$.

Minimizing $M_{\oplus}^2(\beta, y)$

To approximate the $\inf_{y \in Y_h, \beta > 0} M_{\oplus}^2(\beta, y)$ we iterate the following two steps

- Step1: minimizing over $y_h \in Y_h$.
- Step2: minimizing over $\beta > 0$.

Step 1: Minimizing over $y_h \in Y_h$

Since it is easier we minimize $M_{\oplus}^2(\beta, y)$ instead of $M_{\oplus}(\beta, y)$. This is done by computing the Gateaux derivative $(M_{\oplus}^2(y))'(\tilde{y})$ for some arbitrary function $\tilde{y} \in H(div, \Omega)$ and find y such that

$$(M_{\oplus}^2(y))'(\tilde{y}) = 0 \quad \forall \tilde{y} \in Y.$$

By using this we end up in

$$a_1 \int_{\Omega} (A^{-1} y).\tilde{y} \mathrm{d}\boldsymbol{x} + a_2 \int_{\Omega} div(y) div(\tilde{y}) \mathrm{d}\boldsymbol{x} = a_1 \int_{\Omega} \nabla u_h.\tilde{y} \mathrm{d}\boldsymbol{x} + a_2 \int_{\Omega} f.div(\tilde{y}) \mathrm{d}\boldsymbol{x}$$

for all $\tilde{y} \in Y$.

If we approximate this solution in a finite dimensional subspace Y_h we end up in a linear system

$$L_h y_h = r_h,$$

where L_h can be written as

$$L_h = a_1 L_h^1 + a_2 L_h^2,$$

and r_h as

$$r_h = a_1 r_h^1 + a_h^2.$$

If we use this property we do not have to assemble r_h and L_h in every step since we can just compute this linear combination. However this step is very costly.

Step 2: Minimizing over $\beta > 0$

In this case we can simply use minimization for real numbers. This leads to the choice of β as

$$\beta = C_\Omega \sqrt{\frac{B_1}{B_2}}.$$

The evaluation of B_1 and B_2 is cheap, since they are integral evaluations, Step 2 is rather cheap compared which the costs of Step 1.

The Minimization Algorithm

Input: f, u_h, C_Ω
Output: $M_\oplus$

- $\beta =$ initial guess
- Assemble $L_h^1, L_h^2, r_h^1, r_h^2$
- **while** convergence criteria is not fulfilled (and $Iter < MaxIter$)
  - Step 1:
    - $L_h = (1+\beta)L_h^1 + (1+\frac{1}{\beta})C_\Omega^2 L_h^2$
    - $r_h = (1+\beta)r_h^1 + (1+\frac{1}{\beta})C_\Omega^2 r_h^2$
    - Solve: $L_h y_h = r_h$ to obtain y_h
  - Step 2:
    - $B_1 = \|Au_h - y_h\|_{A^{-1}}^2$
    - $B_2 = \|div(y_h) + f\|_{L^2(\Omega)}^2$
    - $\beta = C_\Omega \sqrt{\frac{B_1}{B_2}}$
- **end while**
- $M_\oplus = \sqrt{(1+\beta)B_1 + (1+\frac{1}{\beta})C_\Omega^2 B_2}$

Local Error Estimator 2

Since we know that $a_1 B_1 \to \|\nabla u - \nabla u_h\|_A^2$ and $a_2 B_2 \to 0$ we use the local error estimate

$$\eta_Q^2 := \int_Q (\nabla u_h - A^{-1} y_h).(A \nabla u_h - y_h) \mathrm{d}\boldsymbol{x},$$

to estimate the local error in the cell Q. Now we can again mark the cells with biggest error and refine them afterwards. The error distribution of this estimator is captured correctly if

$$a_1 B_1 > C_\oplus a_2 B_2$$

for some $C_\oplus > 1$. Numerical examples showed that the error indicator $I_{eff} := \frac{\sqrt{a_1 B_1}}{\|\nabla u - \nabla u_h\|_A}$ has a similar behaviour as $\sqrt{1 + \frac{1}{C_\oplus}}$.

Numerical examples

Example 1: In this example we consider $\Omega = (0,1)^2$ and let f, g_D be chosen such that

$$u(x,y) = sin(6\pi x)sin(3\pi y).$$

Here we use the Spline space $V_h := \mathcal{S}_h^{2,2}$

Comparison of different spaces Y_h

For the example we will consider the following three options for the choice of $Y_h = \hat{Y}_h \circ G^{-1}$ where G denotes the geometric transformation.

- Case 0: $\hat{Y}_h = \mathcal{S}_h^{p+1,p} \otimes \mathcal{S}_h^{p,p+1}$ (here Y_h is defined via the Piola transform)
- Case 1: $\hat{Y}_h = \mathcal{S}_h^{p+1,p+1} \otimes \mathcal{S}_h^{p+1,p+1}$
- Case 2: $\hat{Y_{Kh}} = \mathcal{S}_{Kh}^{p+K,p+K} \otimes \mathcal{S}_{Kh}^{p+K,p+K}$ for $K = 2$
- Case 3: $\hat{Y_{Kh}} = \mathcal{S}_{Kh}^{p+K,p+K} \otimes \mathcal{S}_{Kh}^{p+K,p+K}$ for $K = 4$

Example 1, Case 0

Table 1

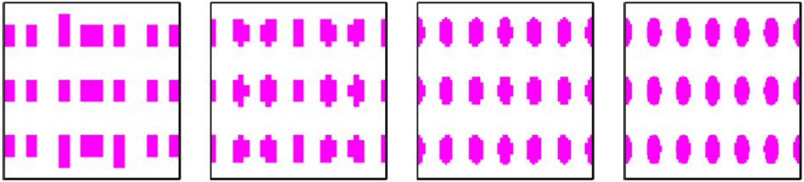
Efficiency index and components of the majorant in Example 1, Case 0, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_h^{3,2} \otimes \mathcal{S}_h^{2,3}$.

Mesh-size	I_{eff}	$a_1 B_1$	$a_2 B_2$	$C_{\oplus}$
8×8	3.43	2.62e+01	1.17e+02	0.2
16×16	1.92	6.07e−01	6.19e−01	1.0
32×32	1.41	2.29e−02	9.71e−03	2.4
64×64	1.20	1.15e−03	2.33e−04	4.9
128×128	1.10	6.51e−05	6.54e−06	10.0
256×256	1.05	3.87e−06	1.95e−07	19.8
512×512	1.03	2.36e−07	5.94e−09	39.7

Screenshot taken from the paper [Kleiss &Tomar 2015]

Table 2
Number of DOF and timings in Example 1, Case 0, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_h^{3,2} \otimes \mathcal{S}_h^{2,3}$.

Mesh-size	#DOF		Assembling-time			Solving-time			Sum		
	u_h	y_h	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$
8×8	100	220	0.04	0.17	4.39	<0.01	<0.01	5.16	0.04	0.17	4.40
16×16	324	684	0.14	0.59	4.25	<0.01	0.01	5.39	0.14	0.60	4.26
32×32	1 156	2 380	0.46	2.17	4.70	0.01	0.03	4.71	0.47	2.20	4.70
64×64	4 356	8 844	1.82	8.51	4.68	0.03	0.20	6.15	1.85	8.70	4.70
128×128	16 900	34 060	7.38	34.19	4.63	0.15	0.87	5.70	7.54	35.06	4.65
256×256	66 564	133 644	33.30	149.78	4.50	0.84	5.66	6.78	34.14	155.44	4.55
512×512	264 196	529 420	191.11	766.10	4.01	3.77	33.92	9.00	194.88	800.03	4.11



(a) 16×16. (b) 32×32. (c) 64×64. (d) 128×128.

Fig. 4. Cells marked by error estimator with $\psi = 20\%$ in Example 1, Case 0, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_h^{3,2} \otimes \mathcal{S}_h^{2,3}$.

Screenshot taken from the paper [Kleiss &Tomar 2015]

Example 1 Case 1

Table 3
Efficiency index and components of the majorant in Example 1, Case 1, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_h^{3,3} \otimes \mathcal{S}_h^{3,3}$.

Mesh-size	I_{eff}	$a_1 B_1$	$a_2 B_2$	$C_\oplus$
8×8	2.77	8.08e+01	1.24e+01	6.5
16×16	1.71	5.75e−01	3.96e−01	1.5
32×32	1.32	2.14e−02	7.05e−03	3.0
64×64	1.16	1.11e−03	1.78e−04	6.2
128×128	1.08	6.39e−05	5.08e−06	12.6
256×256	1.04	3.83e−06	1.53e−07	25.0
512×512	1.02	2.35e−07	4.69e−09	50.1

Table 4
Number of DOF and timings in Example 1, Case 1, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_h^{3,3} \otimes \mathcal{S}_h^{3,3}$.

Mesh-size	#DOF		Assembling-time			Solving-time			Sum		
	u_h	y_h	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$
8×8	100	242	0.04	0.11	2.78	<0.01	<0.01	1.51	0.04	0.11	2.76
16×16	324	722	0.12	0.34	2.86	<0.01	0.01	5.33	0.12	0.35	2.90
32×32	1 156	2 450	0.46	1.35	2.94	0.01	0.05	7.69	0.47	1.40	3.01
64×64	4 356	8 978	1.77	5.30	2.99	0.03	0.27	8.02	1.80	5.57	3.09
128×128	16 900	34 322	7.39	21.89	2.96	0.16	1.45	9.26	7.55	23.34	3.09
256×256	66 564	134 162	33.00	94.69	2.87	0.84	8.83	10.54	33.84	103.52	3.06
512×512	264 196	530 450	191.59	498.20	2.60	3.83	61.45	16.06	195.42	559.65	2.86

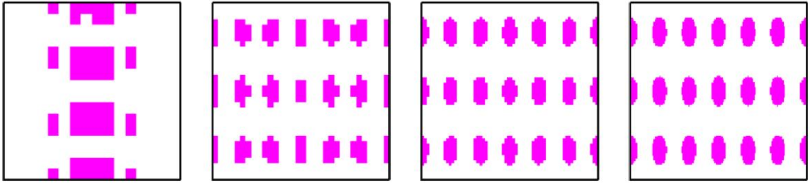
Screenshot taken from the paper [Kleiss &Tomar 2015]

Example 1 Case 2

Table 5
Efficiency index and components of the majorant in Example 1, Case 2, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_{2h}^{4,4} \otimes \mathcal{S}_{2h}^{4,4}$.

Mesh-size	I_{eff}	$a_1 B_1$	$a_2 B_2$	$C_{\oplus}$
8×8	14.19	1.59e+03	8.53e+02	1.9
16×16	8.49	1.97e+01	4.32e+00	4.6
32×32	1.82	3.05e−02	2.41e−02	1.3
64×64	1.16	1.12e−03	1.76e−04	6.4
128×128	1.04	6.14e−05	2.24e−06	27.4
256×256	1.01	3.72e−06	3.32e−08	112.0
512×512	1.00	2.31e−07	5.13e−10	450.3

Screenshot taken from the paper [Kleiss &Tomar 2015]



(a) 16 × 16. (b) 32 × 32. (c) 64 × 64. (d) 128 × 128.

Fig. 6. Cells marked by error estimator with $\psi = 20\%$ in Example 1, Case 2, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_{2h}^{4,4} \otimes \mathcal{S}_{2h}^{4,4}$.

Table 6

Number of DOF and timings in Example 1, Case 2, $\hat{V}_h = \mathcal{S}_h^{2,2}$, $\hat{Y}_h = \mathcal{S}_{2h}^{4,4} \otimes \mathcal{S}_{2h}^{4,4}$.

Mesh-size	#DOF		Assembling-time			Solving-time			Sum		
	u_h	y_h	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$	*pde*	*est*	$\frac{est}{pde}$
8 × 8	100	128	0.03	0.05	1.39	<0.01	<0.01	1.16	0.04	0.05	1.39
16 × 16	324	288	0.14	0.18	1.29	<0.01	<0.01	0.92	0.14	0.18	1.28
32 × 32	1 156	800	0.54	0.59	1.10	0.01	0.02	2.32	0.55	0.61	1.11
64 × 64	4 356	2 592	1.91	2.33	1.22	0.04	0.08	2.09	1.95	2.40	1.23
128 × 128	16 900	9 248	7.46	9.54	1.28	0.19	0.51	2.75	7.64	10.05	1.32
256 × 256	66 564	34 848	33.93	39.02	1.15	0.90	2.59	2.88	34.82	41.60	1.19
512 × 512	264 196	135 200	196.23	177.98	0.91	4.08	15.91	3.90	200.31	193.89	0.97

Screenshot taken from the paper [Kleiss &Tomar 2015]

Example 6

In this example $\Omega = (0,1)^2$ and let f and g be chosen such that the exact solution is given by the function

$$u = (x^2 - x)(y^2 - y)e^{-100\|(x,y)-(0.8,0.05)\|_{\ell_2}^2 - 100\|(x,y)-(0.8,0.05)\|_{\ell_2}^2}$$

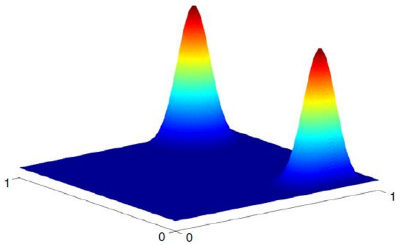
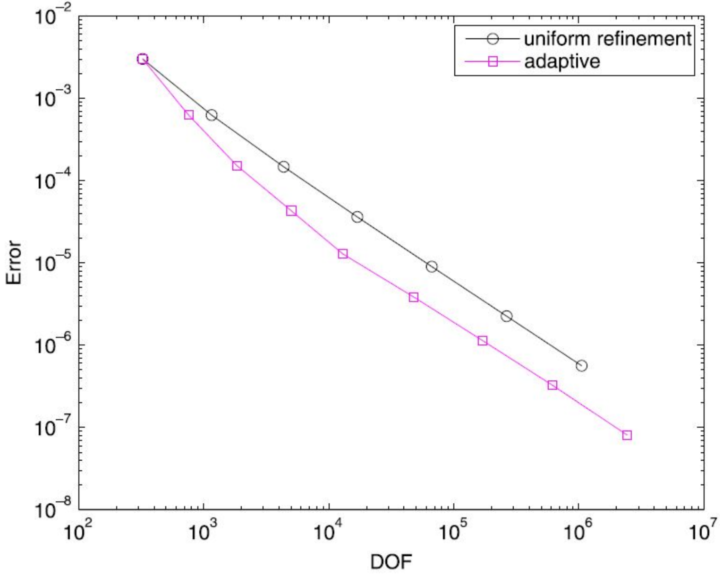
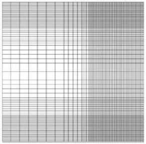


Fig. 13. Exact solution, Example 6.

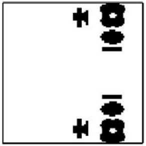
Screenshot taken from the paper [Kleiss &Tomar 2015]



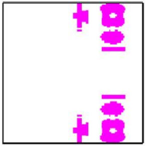
Screenshot taken from the paper [Kleiss &Tomar 2015]



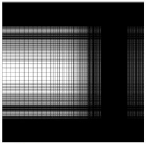
(a) Mesh 4.



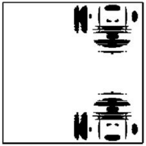
(b) Cells marked by exact error on mesh 4.



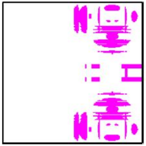
(c) Cells marked by estimator on mesh 4.



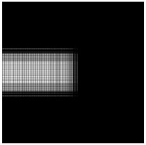
(d) Mesh 7.



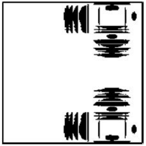
(e) Cells marked by exact error on mesh 7.



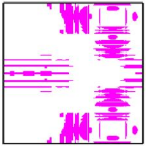
(f) Cells marked by estimator on mesh 7.



(g) Mesh 9.



(h) Cells marked by exact error on mesh 9.



(i) Cells marked by estimator on mesh 9.

Screenshot taken from the paper [Kleiss &Tomar 2015]

Conclusions

- We presented a local error estimator for isogeometric analysis with a guaranteed upper bound.
- This local error estimator captures the region for refinement similar than the exact local error.
- The increase of the polynomial degree in the space Y_h does increase the DOFs just slightly if we compare it to FEM.

Thank you for your attention!