Isogeometric Analysis for Maxwell's equation

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Section 1

Introduction

Maxwell's Equation

$$\frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \qquad \text{(Faraday's law)}$$
$$\frac{\partial D}{\partial t} - \operatorname{curl} H = -J, \qquad \text{(Ampère's law)}$$
$$\operatorname{div} D = \rho, \qquad \text{(electrical Gauss law)}$$
$$\operatorname{div} B = 0, \qquad \text{(magnetic Gauss law).}$$

Material laws

$$D = \epsilon E,$$
$$B = \nu(H + M),$$
$$J = \sigma E.$$

Quantities

B .. mag. field	J .. el. current
E .. el. field	ρ .. el. charge density
D .. el. displacement	ϵ .. el. permittivity
H .. mag. induction	ν .. mag. permeability
M .. magnetization	σ .. conductivity

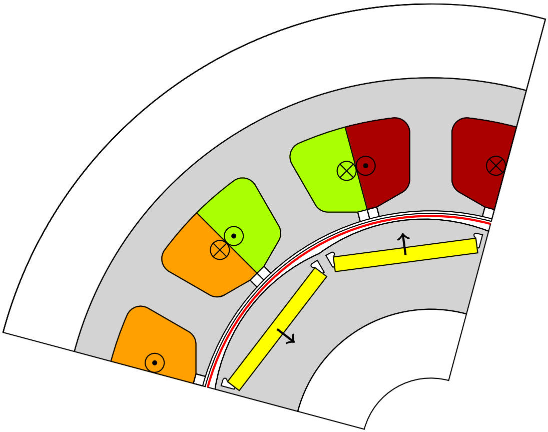

Figure: Motor sector

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A. Buffa, G. Sangalli, and R. Vázquez.
Isogeometric analysis in electromagnetics: B-splines approximation.
Computer Methods in Applied Mechanics and Engineering, 199(17–20):1143 – 1152, 2010.

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An arbitrary high-order spline finite element solver for the time domain maxwell equations.
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R. Vazquez and A. Buffa.
Isogeometric analysis for electromagnetic problems.
IEEE Transactions on Magnetics, 46(8):3305–3308, Aug 2010.

Eigenvalue Problem

Find $\omega \in \mathbb{R}$ and $u \in H_0(\mathrm{curl};\Omega)$ such that

$$\int_\Omega \mu^{-1} \,\mathrm{curl}\, u \,\mathrm{curl}\, v = \omega^2 \int_\Omega \epsilon u \cdot v \quad \forall v \in H_0(\mathrm{curl};\Omega).$$

Source Problem

Find $u \in H_0(\mathrm{curl};\Omega)$ such that

$$\int_\Omega \mu^{-1} \,\mathrm{curl}\, u \cdot \mathrm{curl}\, v - \omega^2 \int_\Omega \epsilon u \cdot v = \int_\Omega f \cdot v \quad \forall v \in H_0(\mathrm{curl};\Omega).$$

physical Domain Ω, and Parametrization $\mathbf{F}$

$\Omega \subset \mathbb{R}^3$:	bounded, simply connected Lipschitz domain with
$\partial\Omega$:	connected boundary
$\mathbf{F} : \widehat{\Omega} \to \Omega$:	continuously differentiable geometrical mapping with continuously differentiable inverse

Sobolev spaces

$$\mathbf{H}(\mathbf{curl};\Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) | \, \mathbf{curl}(\mathbf{v}) \in \mathbf{L}^2(\Omega)\right\}$$
$$\mathbf{H}(\mathrm{div};\Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) | \, \mathrm{div}(\mathbf{v}) \in \mathbf{L}^2(\Omega)\right\}$$

De Rham Complex

$$\mathbb{R} \longrightarrow H^1(\widehat{\Omega}) \xrightarrow{\widehat{\mathbf{grad}}} \mathbf{H}(\mathbf{curl}; \widehat{\Omega}) \xrightarrow{\widehat{\mathbf{curl}}} \mathbf{H}(\mathrm{div}; \widehat{\Omega}) \xrightarrow{\widehat{\mathrm{div}}} L^2(\widehat{\Omega}) \longrightarrow 0$$

$$\mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\mathbf{grad}} \mathbf{H}(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} \mathbf{H}(\mathrm{div}; \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow 0$$

Exact for Ω (and $\widehat{\Omega}$) simply connected.

Pullback operators

$$
\begin{aligned}
\iota^0(\phi) &:= \phi \circ \mathbf{F}, & \phi &\in H^1(\Omega) \\
\iota^1(\mathbf{u}) &:= (D\mathbf{F})^T(\mathbf{u} \circ \mathbf{F}), & \mathbf{u} &\in \mathbf{H}(\mathbf{curl};\Omega) \\
\iota^2(\mathbf{v}) &:= det(D\mathbf{F})(D\mathbf{F})^{-1}(\mathbf{v} \circ \mathbf{F}), & \mathbf{v} &\in \mathbf{H}(\mathrm{div};\Omega) \\
\iota^3(\varphi) &:= det(D\mathbf{F})(\varphi \circ \mathbf{F}), & \varphi &\in L^2(\Omega)
\end{aligned}
$$

De Rham Complex

$$\begin{array}{ccccccccc}
\mathbb{R} \longrightarrow H^1(\widehat{\Omega}) & \xrightarrow{\widehat{\mathbf{grad}}} & \mathbf{H}(\mathbf{curl};\widehat{\Omega}) & \xrightarrow{\widehat{\mathbf{curl}}} & \mathbf{H}(\mathrm{div};\widehat{\Omega}) & \xrightarrow{\widehat{\mathrm{div}}} & L^2(\widehat{\Omega}) \longrightarrow 0 \\
\iota^0 \uparrow & & \iota^1 \uparrow & & \iota^2 \uparrow & & \iota^3 \uparrow \\
\mathbb{R} \longrightarrow H^1(\Omega) & \xrightarrow{\mathbf{grad}} & \mathbf{H}(\mathbf{curl};\Omega) & \xrightarrow{\mathbf{curl}} & \mathbf{H}(\mathrm{div};\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \longrightarrow 0
\end{array}$$

Section 2

B-Splines

Notation

$\Sigma := \{\xi_1, ..., \xi_{n+p+1}\}$ p-open knot vector

$\Sigma' := \{\xi_2, ..., \xi_{n+p}\}$

$B_{i,p}(\xi)$ B-spline functions

$S_p(\Sigma) := span\{B_{i,p}, i = 1, .., n\}$

Anchors and Greville Sites

Anchors	Greville sites
$\xi^A := \begin{cases} \xi_{i+\frac{p+1}{2}} & p \text{ odd}, \\ \frac{\xi_{i+\frac{p}{2}}+\xi_{i+\frac{p}{2}+1}}{2} & p \text{ even} \end{cases}$	$\gamma^A = \frac{\xi_{i+1}+\ldots+\xi_{i+p}}{p}$

Multivariate Case

$$\Sigma_i := \{\xi_{i,1}, \dots, \xi_{i,n_i+p_i+1}\}$$
$$\mathcal{A}_{p_1,\dots,p_d}(\Sigma_1, \dots, \Sigma_d) := \mathcal{A}_{p_1}(\Sigma_1) \times \dots \times \mathcal{A}_{p_d}(\Sigma_d)$$
$$B^A_{p_1,\dots,p_d}(\xi) = B^{A_1}_{p_1}(\xi_1) \dots B^{A_d}_{p_d}(\xi_d)$$
$$S_{p_1,\dots,p_d}(\Sigma_1, \dots, \Sigma_d) := span\{B^A_{p_1,\dots,p_d}\}$$

Spline spaces will be high-order extensions of classical low order Nédélec hexahedral finite elements

Discrete Spaces

$$\widehat{X}_h^0 := S_{p_1,p_2,p_3}(\Sigma_{1,2,3}),$$
$$\widehat{X}_h^1 := S_{p_1-1,p_2,p_3}(\Sigma_1', \Sigma_{2,3}) \times S_{p_1,p_2-1,p_3}(\Sigma_1, \Sigma_2', \Sigma_3) \times S_{p_1,p_2,p_3-1}(\Sigma_{1,2}, \Sigma_3'),$$
$$\widehat{X}_h^2 := S_{p_1,p_{2,3}-1}(\Sigma_1, \Sigma_{2,3}') \times S_{p_1-1,p_2,p_3-1}(\Sigma_1', \Sigma_2, \Sigma_3') \times S_{p_{1,2}-1,p_3}(\Sigma_{1,2}', \Sigma_3),$$
$$\widehat{X}_h^3 := S_{p_1-1,p_2-1,p_3-1}(\Sigma_1', \Sigma_2', \Sigma_3').$$

Discrete De Rham complex

$$\mathbb{R} \longrightarrow \widehat{X}_h^0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{X}_h^1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{X}_h^2 \xrightarrow{\widehat{\text{div}}} \widehat{X}_h^3 \longrightarrow 0$$

This sequence is exact.

Spaces on the Physical domain

Discrete De Rham complex

$$
\begin{array}{ccccccccc}
\mathbb{R} \longrightarrow \widehat{X}_h^0 & \xrightarrow{\widehat{\mathbf{grad}}} & \widehat{X}_h^1 & \xrightarrow{\widehat{\mathbf{curl}}} & \widehat{X}_h^2 & \xrightarrow{\widehat{\mathrm{div}}} & \widehat{X}_h^3 & \longrightarrow 0 \\
\iota^0 \uparrow & & \iota^1 \uparrow & & \iota^2 \uparrow & & \iota^3 \uparrow & \\
\mathbb{R} \longrightarrow X_h^0 & \xrightarrow{\mathbf{grad}} & X_h^1 & \xrightarrow{\mathbf{curl}} & X_h^2 & \xrightarrow{\mathrm{div}} & X_h^3 & \longrightarrow 0
\end{array}
$$

Discrete Spaces

$$
\begin{aligned}
X_h^0 &:= \{\phi : \iota^0(\phi) \in \widehat{X}_h^0\}, \\
X_h^1 &:= \{\mathbf{u} : \iota^1(\mathbf{u}) \in \widehat{X}_h^1\}, \\
X_h^2 &:= \{\mathbf{v} : \iota^2(\mathbf{v}) \in \widehat{X}_h^2\}, \\
X_h^3 &:= \{\varphi : \iota^3(\varphi) \in \widehat{X}_h^3\}.
\end{aligned}
$$

Projectors onto the B-spline space

$$\widehat{\Pi}^{p_1,p_2,p_3}\phi := \sum_{i_1=2,i_2=2,i_3=2}^{n_1-1,n_2-1,n_3-1} (\lambda_{i_1,i_2,i_3}\phi) B_{i_1,i_2,i_3}$$

where λ_i^p are the dual basis functionals in each variable:

$$\lambda_i^p B_j^p = \delta_{ij}$$

Spline preserving property

$$\begin{aligned}
\widehat{\Pi}^0 \widehat{\phi}_h &:= \widehat{\phi}_h, & \forall \widehat{\phi}_h \in \widehat{X}_h^0 \\
\widehat{\Pi}^1 \widehat{\mathbf{u}}_h &:= \widehat{\mathbf{u}}_h, & \forall \widehat{\mathbf{u}}_h \in \widehat{X}_h^1 \\
\widehat{\Pi}^2 \widehat{\mathbf{v}}_h &:= \widehat{\mathbf{v}}_h, & \forall \widehat{\mathbf{v}}_h \in \widehat{X}_h^2 \\
\widehat{\Pi}^3 \widehat{\varphi}_h &:= \widehat{\varphi}_h, & \forall \widehat{\varphi}_h \in \widehat{X}_h^3
\end{aligned}$$

Projectors onto the B-spline space

$$
\begin{array}{ccccccc}
\mathbb{R} \longrightarrow H^1(\widehat{\Omega}) & \xrightarrow{\widehat{\mathbf{grad}}} & \mathbf{H}(\mathbf{curl};\widehat{\Omega}) & \xrightarrow{\widehat{\mathbf{curl}}} & \mathbf{H}(\mathrm{div};\widehat{\Omega}) & \xrightarrow{\widehat{\mathrm{div}}} & L^2(\widehat{\Omega}) \longrightarrow 0 \\
\widehat{\Pi}_0^0 \downarrow & & \widehat{\Pi}_0^1 \downarrow & & \widehat{\Pi}_0^2 \downarrow & & \widehat{\Pi}_0^3 \downarrow \\
\mathbb{R} \longrightarrow \widehat{X}_h^0 & \xrightarrow{\widehat{\mathbf{grad}}} & \widehat{X}_h^1 & \xrightarrow{\widehat{\mathbf{curl}}} & \widehat{X}_h^2 & \xrightarrow{\widehat{\mathrm{div}}} & \widehat{X}_h^3 \longrightarrow 0
\end{array}
$$

The same holds also for the physical domain.

Approximation estimate

Under the assumption $\gamma_d \geq \alpha_d$, $d = 1, 2, 3$, the following estimates hold:

$$\|\phi - \Pi^0\phi\|_{H^l(\Omega)} \leq Ch^{s-l}\|\phi\|_{H^s(\Omega)} \qquad \forall \phi \in H^1 \cap H^s(\Omega),$$
$$0 \leq l \leq s \leq p+1, l \leq \alpha + 1$$
$$\|\mathbf{u} - \Pi^1\mathbf{u}\|_{H^l(\Omega)} \leq Ch^{s-l}\|\mathbf{u}\|_{H^s(\Omega)} \qquad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}; \Omega) \cap \mathbf{H}^s(\Omega),$$
$$0 \leq l \leq s \leq p, l \leq \alpha$$
$$\|\mathbf{v} - \Pi^2\mathbf{v}\|_{H^l(\Omega)} \leq Ch^{s-l}\|\mathbf{v}\|_{H^s(\Omega)} \qquad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega) \cap \mathbf{H}^s(\Omega),$$
$$0 \leq l \leq s \leq p, l \leq \alpha$$
$$\|\varphi - \Pi^3\varphi\|_{H^l(\Omega)} \leq Ch^{s-l}\|\varphi\|_{H^s(\Omega)} \qquad \forall \varphi \in L^2(\Omega) \cap H^s(\Omega),$$
$$0 \leq l \leq s \leq p, l \leq \alpha.$$

Approximation estimate (Energy Norm)

The following inequalities hold for $0 \leq l \leq s \leq p, l \leq \alpha$:

$$\|\phi - \Pi^0\phi\|_{H^{l+1}(\Omega)} \leq Ch^{s-l}\|\phi\|_{H^{s+1}(\Omega)} \qquad \forall \phi \in H^{s+1}(\Omega),$$

$$\|\mathbf{u} - \Pi^1\mathbf{u}\|_{\mathbf{H}^l(\mathbf{curl};\Omega)} \leq Ch^{s-l}\|\mathbf{u}\|_{\mathbf{H}^s(\mathbf{curl};\Omega)} \qquad \forall \mathbf{u} \in \mathbf{H}^s(\mathbf{curl};\Omega),$$

$$\|\mathbf{v} - \Pi^2\mathbf{v}\|_{\mathbf{H}^l(\mathrm{div};\Omega)} \leq Ch^{s-l}\|\mathbf{v}\|_{\mathbf{H}^s(\mathrm{div};\Omega)} \qquad \forall \mathbf{v} \in \mathbf{H}^s(\mathrm{div};\Omega),$$

$$\|\varphi - \Pi^3\varphi\|_{H^l(\Omega)} \leq Ch^{s-l}\|\varphi\|_{H^s(\Omega)} \qquad \forall \varphi \in H^s(\Omega).$$

Multi-patch domain

Conformity across the interface

If the De-Rham complex is fulfilled on each patch:

- trace continuity on X_h^0
- tangential trace continuity on X_h^1
- normal trace continuity on X_h^2
- no continuity on X_h^3

Geometrical Conformity

On each non-empty patch interface Γ the spaces X_{h,k_1}^0 and X_{h,k_2}^0 coincide, as the corresponding bases do.

Section 3

T-Splines

Notation

$$\Sigma_i := \{\xi_{i,1}, ..., \xi_{i,n_i+p_i+1}\}$$
$$\mathcal{A}_{p_1,...,p_d}(\Sigma_1, ..., \Sigma_d) := \mathcal{A}_{p_1}(\Sigma_1) \times ... \times \mathcal{A}_{p_d}(\Sigma_d)$$
$$B^A_{p_1,...,p_d}(\xi) = B[\Sigma^A_1](\xi_1)...B[\Sigma^A_d](\xi_d)$$
$$T_{p_1,...,p_d}(\mathcal{M}) := span\{B^A_{p_1,...,p_d} : A \in \mathcal{A}_{p_1,...,p_d}(\mathcal{M})\}$$

2D De Rham Complex

We have to modify the mesh not only at the boundary but also at the T-junctions.

$$\begin{aligned}
\widehat{Y}_h^0 &:= T_{p,p}(\mathcal{M}^0) \\
\widehat{Y}_h^1 &:= T_{p-1,p}(\mathcal{M}_1^1) \times T_{p,p-1}(\mathcal{M}_2^1) \\
\widehat{Y}_h^{1*} &:= T_{p,p-1}(\mathcal{M}_2^1) \times T_{p-1,p}(\mathcal{M}_1^1) \\
\widehat{Y}_h^2 &:= T_{p-1,p-1}(\mathcal{M}^2)
\end{aligned}$$

De Rham Complex

$$\mathbb{R} \longrightarrow \widehat{Y}_h^0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{Y}_h^1 \xrightarrow{\widehat{\mathrm{curl}}} \widehat{Y}_h^2 \longrightarrow 0$$

$$\mathbb{R} \longrightarrow \widehat{Y}_h^0 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{Y}_h^{1*} \xrightarrow{\widehat{\mathrm{div}}} \widehat{Y}_h^2 \longrightarrow 0$$

3D De Rham Complex

$$
\begin{aligned}
\widehat{X}_h^0 &:= \widehat{Y}_h^0 \times S_p(\Sigma) \\
\widehat{X}_h^1 &:= [\widehat{Y}_h^1 \times S_p(\Sigma)] \times [\widehat{Y}_h^0 \times S_{p-1}(\Sigma')] \\
\widehat{X}_h^2 &:= [\widehat{Y}_h^{1*} \times S_{p-1}(\Sigma')] \times [\widehat{Y}_h^2 \times S_p(\Sigma)] \\
\widehat{Y}_h^3 &:= \widehat{Y}_h^2 \times S_{p-1}(\Sigma')
\end{aligned}
$$

De Rham Complex

$$
\mathbb{R} \longrightarrow \widehat{X}_h^0 \xrightarrow{\widehat{\mathbf{grad}}} \widehat{X}_h^1 \xrightarrow{\widehat{\mathbf{curl}}} \widehat{X}_h^2 \xrightarrow{\widehat{\mathrm{div}}} \widehat{X}_h^3 \longrightarrow 0
$$

Section 4

Examples

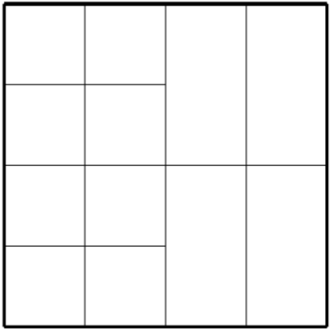
Eigenvalue Problem

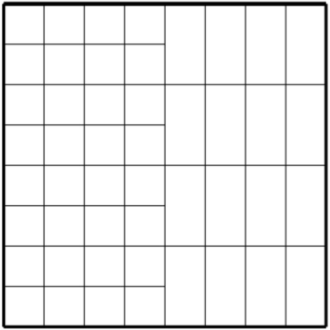
Find $\omega \in \mathbb{R}$ and $u \in H_0(\text{curl}; \Omega)$ such that

$$\int_\Omega \mu^{-1} \, \text{curl}\, u \, \text{curl}\, v = \omega^2 \int_\Omega \epsilon u \cdot v \quad \forall v \in H_0(\text{curl}; \Omega).$$

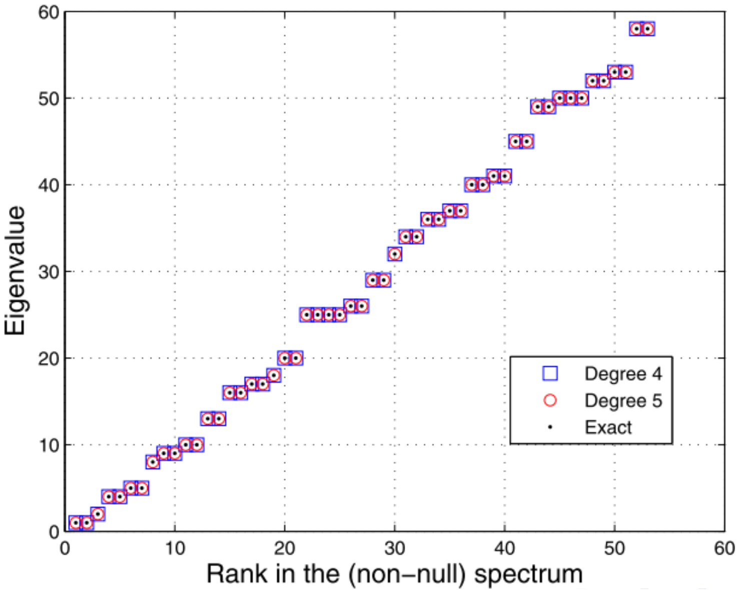
Aim is to show that there are no spurious modes with T-splines.

Results: Square



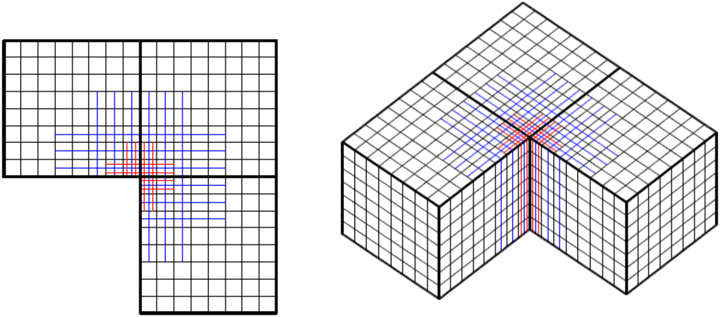


Results: Square

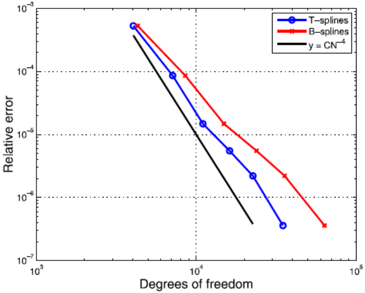


Results: L-Shape Domain 3d

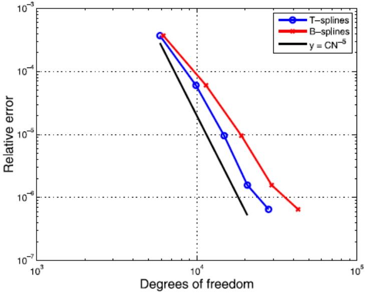
With suitable refinement due to the reentrant edge:



Results: L-Shape Domain 3d



(a) Degree 4



(b) Degree 5

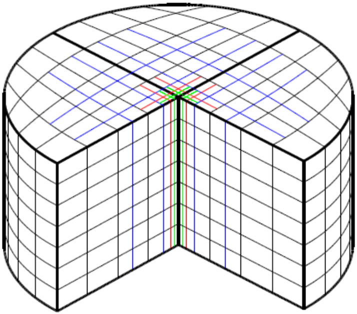
The method is free of spurious modes.

Source Problem

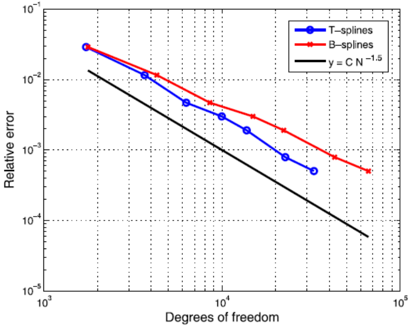
Find $u \in H_0(\text{curl}; \Omega)$ such that

$$\int_\Omega \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v - \omega^2 \int_\Omega \epsilon u \cdot v = \int_\Omega f \cdot v \quad \forall v \in H_0(\text{curl}; \Omega),$$

Results: Three quarters of a cylinder



T- and B-spline of degree 3.



Thank you for your attention!