# T U T O R I A L

## "Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

## Tutorial 10

Date: Thursday, 23 June 2016
Time : $10^{\underline{15}}$ – $11^{\underline{00}}$
Room : K 001A

# 4 Linear Elasticity

## 4.1 The Basic Equations

**28** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary $\Gamma := \partial\Omega$, and let $f \in [L_2(\Omega)]^3$, and $t \in [L_2(\Gamma_t)]^3$. We define the right hand side $F$ of an elastic BVP (see the lectures, Chapter 3, Box (2)) by

$$\langle F\,,\,v\rangle := \int_\Omega f^T\,v\,\mathrm{d}x + \int_{\Gamma_t} t^T\,v\,\mathrm{d}s\,,\quad \forall v \in V_0 = \{v \in V = \left[H^1(\Omega)\right]^3 : v = 0 \text{ on } \Gamma_u\}.$$

Show that $F$ is in $V_0^*$, i.e., $F$ is linear and bounded.

**29** Consider the fixed point iteration (see Lectures, Theorem 3.7, Equation (11)): Given initial guess $u_0 \in V_0$, find $u_{n+1} \in V_0$ such that

$$(u_{n+1}\,,\,v)_1 = (u_n\,,\,v)_1 + \tau\left(\langle F\,,\,v\rangle - a(u_n, v)\right)\,,\quad \forall v \in V_0\,,$$

for $n = 0, 1, 2, \ldots$, where (see also Lectures, Chapter 3, Box (2))

$$\begin{aligned}
(u\,,\,v)_1 &= \int_\Omega \nabla u^T \nabla v\,\mathrm{d}x\,,\\
a(u, v) &= \int_\Omega \varepsilon(u)^T D \varepsilon(v)\,\mathrm{d}x\,,\\
\langle F\,,\,v\rangle &= \int_\Omega f^T v\,\mathrm{d}x + \int_{\Gamma_t} t^T v\,\mathrm{d}s\,.
\end{aligned}$$

Apply a finite element (FE) discretization to this iteration scheme ! Which systems of algebraic equations have to be solved, and which matrix-times-vector multiplications occur within the FE-discretized iteration ?

$\boxed{30^*}$ Show that in the case of the Neumann boundary value problem ($\Gamma_t = \Gamma$) the primal variational problem (2) from the Letcure (Chapter 3) is solvable iff

$$\langle F, v\rangle = 0, \quad \forall v \in \mathcal{R} \text{ (rigid body motions).} \tag{4.38}$$

If this solvability condition is fulfilled, then the solution is uniquely defined up to rigid body motions ! (cf. Exercise 3.8 from the Lecture)

*Hint:* Let us consider the Neumann problem: find $u \in V = [H^1(\Omega)]^3$ :

$$a(u,v) := \int_\Omega \varepsilon(u)^T D\varepsilon(v)\,\mathrm{d}x = \int_\Omega f^T v\,\mathrm{d}x + \int_\Gamma t^T v\,\mathrm{d}s =: \langle F, v\rangle \quad \forall v \in V. \tag{4.39}$$

Introduce the scalar product (Korn's inequality !)

$$[u,v] := \int_\Omega \varepsilon(u)^T D\varepsilon(v)\,\mathrm{d}x + \int_\Omega u^T v\,\mathrm{d}x$$

in $V$, and rewrite (4.39) as

$$[u,v] - \int_\Omega u^T v\,\mathrm{d}x = \langle F, v\rangle =: [\tilde{f}, v] \quad \forall v \in V,$$

where $\tilde{f} = JF \in V$ is uniquely defined by Riesz-isomorphism $J : V^* \to V$ provided that $V$ is now equiped with scalar product $[\cdot,\cdot]$. Define the operator $K : V \to V$:

$$[Ku, v] := \int_\Omega u^T v\,\mathrm{d}x \quad \forall u, v \in V,$$

and show that $K$ is compact ($H^1$ is compactly embedded in $L_2$ !) ! Then (4.39) can be rewritten as operator equation

$$(I - K)u = \tilde{f} \quad \text{in } V. \tag{4.40}$$

Now apply the Fredholm-theory to the operator equation (4.40) !