T U T O R I A L

"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 05

Date: Thursday, 28 April 2016
Time : $10^{\underline{15}}$ – $11^{\underline{00}}$
Room : K 001A

3 Analysis and Numerics of Mixed Variational Problems

3.1 Mixed Variational Problems

Consider the mixed variational problem: Find $u \in X$ and $\lambda \in \Lambda$, such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle f, v\rangle\,, \quad \forall v \in X\,,\\ b(u,\mu) &= \langle g, \mu\rangle\,, \quad \forall \mu \in \Lambda\,. \end{aligned}$$

In order to guarantee a unique existence of the solution (see Theorem 2.4 of *Brezzi* in the lectures) one has to verify the following conditions:

1. The linear forms f and g are continuous, i.e.,

$$f \in X^*\,, \quad g \in \Lambda^*\,, \tag{3.26}$$

2. the bilinear forms $a(\cdot,\cdot) : X \times X \to \mathbf{R}$ and $b(\cdot,\cdot) : X \times \Lambda \to \mathbf{R}$ are continuous, i.e., $\exists$ positive constants α_2, β_2:

$$|a(u,v)| \;\leq\; \alpha_2 \|u\|_X \|v\|_X\,, \quad \forall u, v \in X\,, \tag{3.27}$$

$$|b(v,\mu)| \;\leq\; \beta_2 \|v\|_X \|\mu\|_\Lambda\,, \quad \forall v \in X\,, \forall \mu \in \Lambda\,, \tag{3.28}$$

3. LBB (Ladyshenskaja – Babuska – Brezzi) condition: $\exists$ positive constant β_1:

$$\inf_{\substack{\mu\in\Lambda\\ \mu\neq 0}} \sup_{\substack{v\in X\\ v\neq 0}} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_\Lambda} \geq \beta_1\,, \tag{3.29}$$

4. Ker B–ellipticity, i.e., $\exists$ positive constant α_1:

$$a(v,v) \geq \alpha_1 \|v\|_X^2\,, \quad \forall v \in \operatorname{Ker} B\,, \tag{3.30}$$

where $\operatorname{Ker} B = \{v \in X \mid Bv = 0 \text{ (in } \Lambda^*)\} = \{v \in X \mid \underbrace{b(v,\mu)}_{=\langle Bv\,,\,\mu\rangle} = 0\,, \forall \mu \in \Lambda\}$.

$\boxed{13}$ Consider the mixed formulation of the 1st BVP of the biharmonic equation (see Example 1.3 in the lectures, and Exercise 9 of the tutorials):
Find $w \in X := H^1(\Omega)$ and $u \in \Lambda := H_0^1(\Omega)$ such that there holds

$$
\begin{aligned}
\int_\Omega w\, m \,\mathrm{d}x - \int_\Omega \nabla m \cdot \nabla u \,\mathrm{d}x &= 0\,, \quad \forall m \in X\,,\\
-\int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x \qquad\qquad &= \int_\Omega f\, v \,\mathrm{d}x\,, \quad \forall v \in \Lambda\,,
\end{aligned}
$$

Show that for this problem, the conditions (3.27) and (3.29) are satisfied ! What can you say about (3.30) ?

$\boxed{14}$ Consider the Stokes problem (see Example 1.1 in the lectures): Find $u \in X := [H_0^1(\Omega)]^3$ and $p \in \Lambda := \{q \in L_2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0\}$ such that there holds

$$
\begin{aligned}
\frac{1}{\mathrm{Re}} \int_\Omega \nabla u : \nabla v \,\mathrm{d}x - \int_\Omega \operatorname{div} v\, p \,\mathrm{d}x &= \int_\Omega f\, v \,\mathrm{d}x\,, \quad \forall v \in X\,,\\
-\int_\Omega \operatorname{div} u\, q \,\mathrm{d}x \qquad\qquad &= 0\,, \quad \forall q \in \Lambda\,,
\end{aligned}
$$

where the Reynolds number Re is positive, and where : denotes the inner product $A : B = \sum_{i,j=1}^3 a_{ij}\, b_{ij}$, defined for matrices $A = (a_{ij})_{i,j=1,2,3}$ and $B = (b_{ij})_{i,j=1,2,3}$.
Show that for this problem the conditions (3.27) – (3.30), except for the too difficult part (3.29), are satisfied.

$\boxed{15^*}$ Let X and Λ be real Hilbert spaces and $B : X \to \Lambda^*$ a bounded linear operator. Show that B satisfies the LBB-condition

$$
\exists \beta_1 > 0 : \inf_{\substack{v\in\Lambda \\ v\neq 0}} \sup_{\substack{\tau\in X \\ \tau\neq 0}} \frac{\langle B\tau\,,\, v\rangle}{\|\tau\|_X \|v\|_\Lambda} \geq \beta_1\,,
$$

if and only if there exists a positive constant c such that for all $v^* \in \Lambda^*$ there exists a $\tau \in X$ such that $B\tau = v^*$ and $\|\tau\|_X \leq c\,\|v^*\|_{\Lambda^*}$.

$\boxed{16}$ Show directly (without using Theorem 2.4 of *Brezzi*), that under the assumptions of Theorem 2.4 of *Brezzi* the homogeneous mixed variational problem

$$
\begin{aligned}
a\,(u,v) + b\,(v,\lambda) &= 0 \quad \forall\, v \in X\\
b\,(u,\mu) &= 0 \quad \forall\, \mu \in \Lambda
\end{aligned}
$$

has only the trivial solution $(u,\lambda) = (0,0) \in X \times \Lambda$!