Possible Questions for the Oral Examination
as a Road Map for the Preparation to the Examination

1. Derive and discuss the mixed variational formulation for the Stokes Problem (Example 1.1), the Dirichlet Problem for the Poisson Equation (Example 1.2) and the biharmonic equation (Example 1.3) !

2. The solvability of the variational problem
$$\text{Find } u \in U: \ a(u,v) = \langle f, v\rangle \quad \forall v \in V \tag{1}$$
is discussed in the Bubuška-Aziz-Theorem (= Theorem 1.5). State and prove the sufficient and necessary conditions for the unique solvability of the variational problem (1) !

3. State the conditions under which a unique solution of the Galerkin-Petrov scheme
$$\text{Find } u_h \in U_h: \ a(u_h,v_h) = \langle f, v_h\rangle \quad \forall v_h \in V_h \tag{2}$$
exists, and provide a Cea-like estimate for the discretization error $||u - u_h||_U$ (Theorem.1.6) !

4. Show that the non-linear operator equation
$$\text{Find } u \in V_0: \ A(u) = f \text{ in } V_0^* \tag{3}$$
has a unique solution provided that $f \in V_0^*$ and the non-linear operator $A : V_0 \to V_0^*$ is strongly monotone and Lipschitz-continuous ! Apply this theory to a variational inequality of the form
$$\text{Find } u \in U : a(u, v-u) \geq \langle f, v-u\rangle \quad \forall v \in U. \tag{4}$$
Give at least one example leading to a variational inequality of the form (4) !

5. Reformulate the abstract mixed variational problem: Find $(u,\lambda) \in X \times \Lambda$ such that
$$a(u,v) + b(v,\lambda) = \langle f, v\rangle \quad \forall v \in X \tag{5}$$
$$b(u,\mu) = \langle g, \mu\rangle \quad \forall \mu \in \Lambda \tag{6}$$
as operator equation ! State and prove conditions, which are equivalent to the famous inf-sup-condition (Lemma 2.2) !

6. State and prove Berzzi's Theorem (Theorem 2.4) !

7. Provide the theory (existence, uniqueness, discretization error estimates) for the mixed finite element approximation to the mixed variational formulation (5)–(6) ! How can you check the discrete LBB-condition ?

8. Provide equivalent formulations of the Mixed Variational Problem (5)–(6) for the case of symmetric and positive bilinear form $a(.,.)$ (Section 2.2 of the lectures) !

9. Discuss the (preconditioned) UZAWA- and the ARROW-HURWICZ-Algorithms for solving systems of the form

$$\begin{pmatrix} A & B^\top \\ B & -C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix} \tag{7}$$

arising from the mixed finite element approximation to the mixed variational formulation (5)–(6) ! Which iteration error estimates do you know for the preconditioned UZAWA-Algorithm (Theorem 2.17 and Lemma 2.18) ?

10. Explain the Bramble-Pasciak-Transformation and the Bramble-Pasciak-CG for solving (7) including Theorem 2.21, its proof, and Exercise 2.24 (algorithm and iteration error estimates) !

11. Derive the variational formulation of the linear elasticity problem

$$\begin{aligned} -\mathrm{div}(\sigma) &= f \quad \text{in } \Omega, && (8)\\ \sigma &= D\varepsilon \quad \text{in } \Omega, && (9)\\ \varepsilon &= \varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla)^\top, \quad \text{in } \Omega, && (10)\\ u &= 0 \quad \text{on } \Gamma_u, && (11)\\ \sigma \cdot n &= t \quad \text{on } \Gamma_t && (12) \end{aligned}$$

Discuss the solvability (existence and uniqueness) of the variational linear elasticity problem (Subsection 3.2.1) !

12. Derive mixed variational formulations of the linear elasticity problem following the first and the second Hellinger-Reisner principle (Subsection 3.2.2) ! What do you know about the solvability ?

13. Discuss the case of incompressible and almost incompressible materials and its stable mixed finite element approximation (Subsection 3.2.3) !

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