Variational Inequalities with Monotonicity Consequences for Optimization

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Topic of this talk

Definitions and Preliminaries

Variational Inequalities with Monotonicity

Consequences for Optimization

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Definitions and Preliminaries

Variational Inequalities with Monotonicity

- Convexity of solution
- Existence without monotonicity and boundedness (improving a 2A.1)
- Existence for monotone V.I.
- V.I. with strong monotonicity (property of the solutions)

Consequences for Optimization

- 2nd-Order optimality on polyhedral convex set
- Parametrized minimization problem
- Tilted minimization problem
- Minimization problem with system of constraints (not convex)

Normal Cone and Variational inequality

Let $C \subset \mathbb{R}^n$ be convex, $x \in C$.

Definition (Normal Cone)

$$N_c(x) = \{v \mid \langle v, x' - x\rangle \leq 0, \quad \forall x' \in C\} \text{ (closed and convex)}$$

Normal Cone and Variational inequality

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Definition (Variational inequality)

$$f(x) + N_C(x) \ni 0$$
$$\Leftrightarrow$$
$$-f(x) \in N_C(x)$$
$$\Leftrightarrow$$
$$\langle f(x), x' - x \rangle \geq 0, \quad \forall x' \in C$$

Monotonicity, strong monotonicity and solution mapping

Let $C \subset \mathbb{R}^n$ be convex, $x \in C$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$.

Definition (Monotone and strongly monotone functions)

f is said to be monotone on C if

$$\langle f(x') - f(x), x' - x \rangle \geq 0, \quad \forall x, x' \in C.$$

f is said to be strongly monotone on C with constant $\mu > 0$ if

$$\langle f(x') - f(x), x' - x \rangle \geq \mu |x - x'|^2, \quad \forall x, x' \in C.$$

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Definition (Solution mapping)

The mapping to the perturbation scheme $f(x) - p$ (where p is a parameter vector $p \in \mathbb{R}^n$) is

$$S(p) = \{x \mid p - f(x) \in N_C\} = (f + N_C)^{-1}(p).$$

2F.1: Solution Convexity for Monotone Variational Inequalities

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be monotone in a nonempty closed convex set C. The solution mapping S is closed and convex valued. In particular, the set of solutions (if any) to the variational inequality is ***closed*** *and* ***convex****.*

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Proof.

Since $S(p) = (f_p + N_C)^{-1}(0)$ for $f_p(x) = f(x) - p$ (which is monotone and continuous like f), it is sufficient to deal with $S(0)$. The closedness of $S(0)$ is already shown in **Theorem 2A.1** by W. Stockinger for non-monotone functions (I will state it later). It remains to show convexity on the black board. □

2F.2: Solution Existence for Variational Inequalities without boundedness

This is an extension of **Theorem 2A.1** from W. Stockinger presentation:

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous in a nonempty closed convex set C. Suppose there exist a $\hat{x} \in C$ and $\rho > 0$ such that

$$\text{there is no } x \in C \text{ with } |x - \hat{x}| \geq \rho \text{ and } \langle f(x), x - \hat{x} \rangle \leq 0.$$

Then the variational inequality has a solution and every solution satisfies $|x - \hat{x}| < \rho$.

Proof of Theorem 2F.2

We need the following ingredients which was shown in W. Stockinger presentation:

Theorem (Solutions to Variational Inequalities)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous in a nonempty closed convex set C. The set of solutions to the variational equation is always closed. It is sure to be nonempty when C is bounded.

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Remark

If $C = C_1 \cap C_2$ for closed, convex sets C_1 and C_2 in $\mathbb{R}^n$, then the formula

$$N_c(x) = N_{c_1}(x) + N_{c_2}(x) = \{v_1 + v_2 \mid v_1 \in N_{c_1}(x),\ v_2 \in N_{c_2}(x)\},$$

holds for every $x \in C$ such that there is no $v \neq 0$ with $v \in N_{c_1}(x)$ and $-v \in N_{c_2}(x)$.

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Proof on blackboard!

2F.3: Uniform Local Existence

Corollary

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous in a nonempty closed convex set C. Suppose there exist a $\hat{x} \in C$, $\rho > 0$ and $\eta > 0$ such that

$$\text{there is no } x \in C \text{ with } |x - \hat{x}| \geq \rho \text{ and } \langle f(x), x - \hat{x} \rangle / |x - \hat{x}| \leq \eta.$$

Then the solution mapping S has the property that

$$\emptyset \neq S(v) \subset \{x \in C \mid |x - \hat{x}| < \rho\} \text{ when } |v| \leq \eta.$$

Proof of Corollary 2F.3

Proof.

We show that stronger condition hold for Theorem 2F.2 with $f_v(x) = f(x) - v, \forall |v| < \eta$. Assume that there exists a x s. t. $|x - \hat{x}| \geq \rho$, then from the assumption in the Corollary $\langle f(x), x - \hat{x} \rangle > \eta |x - \hat{x}|$.

We need to show $\langle f(x) - v, x - \hat{x} \rangle > 0$.

$$\begin{aligned} \langle f(x) - v, x - \hat{x} \rangle / |x - \hat{x}| &> \eta - \langle v, x - \hat{x} \rangle / |x - \hat{x}| \\ &> \eta - |v| |x - \hat{x}| / |x - \hat{x}| \\ &= \eta - |v| > 0, \quad \forall x \in C, \end{aligned}$$

$$\text{i.e.} \quad \langle f(x) - v, x - \hat{x} \rangle > (\eta - |v|) \rho > 0.$$

The assumption in the Theorem is assured from the stronger assumption in the Corollary. □

2F.4: Solution Existence for Monotone Variational Inequalities

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and monotone in a nonempty closed convex set C. Let $\hat{x} \in C$ and let W consist of the vectors w with $|w| = 1$ such that $\hat{x} + \tau w \in C$, for all $\tau \in (0, \infty)$, if any.

(a) *If $\lim \tau \to \infty \langle f(\hat{x} + \tau w), w \rangle > 0$ for all $w \in W$, then the solution mapping S is nonempty-valued on a neighborhood of 0.*

(b) *If $\lim \tau \to \infty \langle f(\hat{x} + \tau w), w \rangle = \infty$ for all $w \in W$, then the solution mapping S is nonempty-valued on all of $\mathbb{R}^n$.*

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(b) If $\lim \tau \to \infty \langle f(\hat{x} + \tau w), w \rangle = \infty$ for all $w \in W$, then the solution mapping S is nonempty-valued on all of $\mathbb{R}^n$.

Proof.

The idea of the proof for (a) is to show that limit criteria i (a), which we call A, ensures the assumption in corollary 2F.3, which we call B. This is shown by showing that $!B \to !A$. □

2F.6: Variational Inequalities with Strong Monotonicity

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and strongly monotone with constant $\mu > 0$ in a nonempty closed convex set C. Then the solution mapping S is single-valued on all of $\mathbb{R}^n$ and moreover Lipschitz continuous with constant μ^{-1}.

Proof.

Proof on the blackboard! □

Preliminaries

We look at the variational inequality

$$\nabla g(x) + N_C(x) \ni 0,$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function over a nonempty, closed, convex set $C \subset \mathbb{R}^n$.
Or equivalently

$$\langle \nabla g(x), w \rangle \geq 0 \quad \forall\, w \in T_C(x),$$

where $T_C(x)$ is the tangent cone.

We recall the critical cone from Peter's presentation:

$$K_C(x, -\nabla g(x)) = \{w \in T_C(x) \mid \langle \nabla g(x), w \rangle = 0\},$$

2G.1: Second-Order Optimality on a Polyhedral Convex Set

Theorem

Let C be a polyhedral convex set in $\mathbb{R}^n$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable on C. Let $\bar{x} \in C$ and $\bar{v} = -\nabla g(\bar{x})$.

- (a) *(necessary condition) If g has a local minimum with respect to C at $\bar{x}$, then $\bar{x}$ satisfies the variational inequality and has $\langle w, \nabla^2 g(\bar{x}) w \rangle \geq 0$ for all $w \in K_C(\bar{x}, \bar{v})$.*
- (b) *(sufficient condition) If $\bar{x}$ satisfy the variational inequality and has $\langle w, \nabla^2 g(\bar{x}) w \rangle > 0$ for all nonzero $w \in K_C(\bar{x}, \bar{v})$, then g has a local minimum relative to C at $\bar{x}$, indeed a strong local minimum i the sense of there being an $\epsilon > 0$ such that*

$$g(x) \geq g(\bar{x}) + \epsilon |x - \bar{x}|^2 \quad \forall x \in C \text{ near } \bar{x}.$$

Proof of Theorem 2G.1

Proof.

We will prove the necessary condition on the blackboard!

□

Proof of Theorem 2G.1

Proof.

We will prove the necessary condition on the blackboard!

For the sufficient condition (b) the 1D case is inadequate, since we need a neighborhood of $\bar{x}$ relative to C.
To show (b) one use the 2nd-order Taylor expansion of g at $\bar{x}$.
Furthermore one use the tangent cone property is 2E.3, which requires C is *polyhedral*.

□

Stationary Points

Definition

An x satisfying the variational inequality, will be called a stationary point of g with respect to minimizing over C, regardless of whether or not it furnishes a local or global minimum.

Stationary point mapping for perturbations

We now look at the parametrized problem on the form

$$\text{minimize } g(p,x) \text{ over all } x \in C,$$

where $g : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable w.r.t. x, and C is a nonempty, closed, convex subset of $\mathbb{R}^n$.

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The variational inequality

$$\nabla_x g(p, x) + N_C(x) \ni 0,$$

provides for each p a first-order condition which x must satisfy if it furnishes a local minimum. It describes the stationary points.

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Definition

The stationary point mapping, $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$, is defined by

$$S(p) = \{x \mid \nabla_x g(p, x) + N_C(x) \ni 0\}$$

Stationary point mapping for tilt perturbations

We also look at the auxiliary problem with parameter $v \in \mathbb{R}^n$, in which $g(\bar{p}, \cdot)$ is replaced with it's 2nd-order expansion at $\bar{x}$:

$$\text{minimize } \bar{g}(w) - \langle v, w \rangle \text{ over all } w \in W, \text{ where}$$

$$\bar{g}(w) = g(\bar{p}, \bar{x}) + \langle \nabla_x g(\bar{p}, \bar{x}), w \rangle + \frac{1}{2} \left\langle w, \nabla^2_{xx} g(\bar{p}, \bar{x}) w \right\rangle,$$

$$W = \{w \mid \bar{x} + w \in C\} = C - \bar{x}.$$

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The variational inequality

$$\nabla_x g(\bar{p}, \bar{x}) + \nabla^2_{xx} g(\bar{p}, \bar{x}) w - v + N_W(w) \ni 0,$$
$$\text{where } N_W(w) = N_C(\bar{x} + w)$$

describes the stationary points.

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Definition

The stationary point mapping, $\bar{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, is defined by

$$\bar{S}(v) = \left\{ w \mid \nabla_x g(\bar{p}, \bar{x}) + \nabla^2_{xx} g(\bar{p}, \bar{x}) w + N_W(w) \ni v \right\}.$$

2G.2: Parametric Minimization Over a Convex Set

Theorem (part 1)

Using the notation from the two previous slides, with $\bar{x} \in S(\bar{p})$, that

(a) $\nabla_x g$ is strictly differentiable at $(\bar{p}, \bar{x})$.

(b) $\bar{S}$ has a Lipschitz continuous single-valued localization $\bar{s}$ around 0 for 0.

Then S has a Lipschitz continuous single-valued localization s around $\bar{p}$ for $\bar{x}$ with

$$lip(s; \bar{p}) \leq lip(\bar{s}; 0) \, |\nabla^2_{xp} g(\bar{p}, \bar{x})|$$

and s has a first-order approximation η at $\bar{p}$ given by

$$\eta(p) = \bar{x} + \bar{s}\left(-\nabla^2_{xp} g(\bar{p}, \bar{x})(p - \bar{p})\right).$$

(b) is necessary for S to have a Lipschitz continuous single-valued localization around $\bar{p}$ for $\bar{x}$ when the $n \times d$ matrix $\nabla^2_{xp} g(\bar{p}, \bar{x})$ has rank n.

2G.2: Parametric Minimization Over a Convex Set

Theorem (part 2)

If additionally assuming that C is a polyhedral, condition (b) is equivalent to the condition that, for the critical cone $K = K_C(\bar{x}, -\nabla_x g(\bar{p}, \bar{x}))$, the mapping

$$v \mapsto \bar{S}_0(v) = \left\{ w \mid \nabla^2_{xx} g(\bar{p}, \bar{x})\, w + N_K(w) \ni v \right\}$$

is everywhere single-valued.

Moreover, a sufficient condition for this can be expressed in terms of the critical subspaces $K_C^+(\bar{x}, \bar{v}) = K_C(\bar{x}, \bar{v}) - K_C(\bar{x}, \bar{v})$ and $K_C^-(\bar{x}, \bar{v}) = K_C(\bar{x}, \bar{v}) \cap [-K_C(\bar{x}, \bar{v})]$ for $v = -\nabla_x g(\bar{p}, \bar{x})$:

$$\left\langle w, \nabla^2_{xx} g(\bar{p}, \bar{x})\, w \right\rangle > 0,$$

for every nonzero $w \in K_C^+(\bar{x}, \bar{v})$ with $\nabla^2_{xx} g(\bar{p}, \bar{x})\, w \perp K_C^-(\bar{x}, \bar{v})$.

Proof of Theorem 2G.2 (part 1)

We recall Theorem 2E.1 from Peter's presentation:

Theorem (2E.1)

For a variational inequaility and its solution mapping, let $\bar{p}$ and $\bar{x}$ be such that $\bar{x} \in S(\bar{p})$. Assume that

(a) *f is strictly differentiable at $(\bar{p}, \bar{x})$;*

(b) *the inverse G^{-1} of the mapping*

$$G(x) = f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x})(x - \bar{x}) + N_C(x), \quad \text{with } G(\bar{x}) \ni 0,$$

has a Lipschitz continuous single-valued localization σ around 0 for $\bar{x}$.

Then S has a Lipschitz continuous single-valued localization s around $\bar{p}$ for $\bar{x}$ with

$$lip(s; \bar{p}) \leq lip(\sigma; 0) \cdot |\nabla_p f(\bar{p}, \bar{x})|,$$

and this localization s has a first-order approximation η at $\bar{p}$ given by

$$\eta(p) = \sigma(-\nabla_p f(\bar{p}, \bar{x})(p - \bar{p})).$$

Proof of Theorem 2G.2 (part 2)

Theorem (2E.1 (cont'd))

Moreover, under the ample parametrization condition

$$\text{rank}\,\nabla_p f(\overline{p}, \overline{x}) = n,$$

the existence of a Lipschitz continuous single-valued localization s of S around $\overline{p}$ for $\overline{p}$ not only follows from, but also necessitates the existence of a localization σ of G^{-1} having the properties described.

Proof.

We apply this Theorem with $f(x, p) = \nabla_x g(p, x)$ □

2G.3 Stability of a Local Minimum on Polyhedral Convex Set

Theorem

In the setting of the parametrized minimization problem (not the auxiliary tilted problem) and its stationary point mapping S we assume that C is polyhedral and that $\nabla_x g(p,x)$ is strictly differentiable with respect to (p,x) at $(\bar{p},\bar{x})$, where $\bar{x} \in S(\bar{p})$. W.r.t. the critical subspace $K_C^+(\bar{x},\bar{v})$ for $\bar{v} = -\nabla_x g(\bar{p},\bar{x})$, assume that

$$\left\langle w, \nabla_{xx}^2 g(\bar{p},\bar{x})\, w \right\rangle > 0, \ \text{for every nonzero } w \in K_C^+(\bar{x},\bar{v}).$$

Then S has a localization s with the properties from Theorem 2G.2 and also with the property that for every p in some neighborhood of $\bar{p}$, the point $x = s(p)$ furnishes a strong local minimum. Moreover the assumption is necessary for the existence of a localization s with all there properties, when the $n \times d$ matrix $\nabla_{xp}^2 g(\bar{p},\bar{x})$ has rank n.

2G.4: Tilted Minimization of Strongly Convex Functions

Proposition

Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable on an open set O and let g be strongly convex with constant $\mu > 0$ on $C \subset O$, where C is a nonempty, closed, convex set. Then for every $v \in \mathbb{R}^n$ the problem

$$\text{minimize } g(x) - \langle v, x \rangle \text{ over } x \in C$$

has a unique solution $s(v)$, and the solution mapping s is a Lipschitz continuous function on $\mathbb{R}^n$ with constant μ^{-1}.

Proof of Proposition 2G.4

We need the **Theorem 2A.7** which was shown in W. Stockinger presentation:

Theorem (Basic Variational Inequality for Minimization)

Let $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable on an open convex set O and let C is a nonempty, closed, convex subset of O. In minimizing g over C, the variation inequality

$$\nabla g(x) + N_C(x) \ni 0,$$

is necessary for x to furnish a local minimum. It is both necessary and sufficient for a global minimum if g is convex.

And we need the lemma:

Lemma (shown in exercise 2A.6 (a))

Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable on an open set O and let g be strongly convex with constant $\mu > 0$ on $C \subset O$, then the function $f(x) = \nabla g(x)$ strongly monotone on C with constant μ. This is an if and only if condition.

Minimization with system of constraints

The set C is now not necessary convex!

We look at the nonlinear programming problem:

$$\text{minimize } g_0(x) \text{ over all } x \text{ satisfying } g_i(x) \begin{cases} \leq 0 & \text{for } i \in [1, s], \\ = 0 & \text{for } i \in [s+1, m]. \end{cases}$$

We assume that the functions $g_0, g_1, \ldots, g_m$ are twice continuously differentiable on $\mathbb{R}^n$.

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We assume that the functions $g_0, g_1, \ldots, g_m$ are twice continuously differentiable on $\mathbb{R}^n$. From **Theorem 2A.9** we know that there exists a multiplier vector $y = (y_1, \ldots, y_m)$ relative to x, fulfilling the Karush-Kuhn-Tucker (KKT) conditions:

$$y \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s},\ g_i(x) \begin{cases} \leq 0 & \text{for } i \in [1, s], \text{ with } y_i = 0, \\ = 0 & \text{for all other } i \in [1, m], \end{cases}$$

$$\nabla g_0(x) + y_1 \nabla g_1(x) + \ldots + y_m \nabla g_m(x) = 0.$$

Lagrangian function

We want to take a second-order approach to local sufficiency, hence we rewrite the problem in terms of the Lagrangian.

From **Theorem 2A.10** we know that the KKT formulation on the previous slide can identified in terms of the Lagrangian function

$$L(x, y) = g_0(x) + y_1 g_1(x) + \ldots + y_m g_m(x)$$

with the V.I. for a continuously differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ and a polyhedral convex cone $E \subset \mathbb{R}^n \times \mathbb{R}^m$,

$$f(x, y) + N_E(x, y) \ni (0, 0),$$

$$\text{where} \begin{cases} f(x, y) & = (\nabla_x L(x, y) - \nabla_y L(x, y))^T, \\ E & = \mathbb{R}^n \times \left[\mathbb{R}^s_+ \times \mathbb{R}^{m-s}\right]. \end{cases}$$

2G.6: Second-Order Optimality in Nonlinear Programming

Theorem

Let $\bar{x}$ be a point satisfying the constrains, $g_i(\bar{x})$. Let $I(\bar{x})$ be the set of indices i of the active constraints at $\bar{x}$, and assume that the gradients $\nabla g_i(\bar{x})$ for $i \in I(\bar{x})$ are linear independent. Let K consists of the vector $w \in \mathbb{R}^n$ satisfying

$$\langle \nabla g_i(\bar{x}), w \rangle \begin{cases} \leq 0 & \text{for } i \in I(\bar{x}) \text{ with } i \leq s, \\ = 0 & \text{for all other } i \in I(\bar{x}) \text{ and also for } i = 0. \end{cases}$$

(a) *(Necessary Condition) If $\bar{x}$ furnishes a local minimum in the minimization problem, then a multiplier vector $\bar{y}$ exist such that $(\bar{x}, \bar{y})$ satisfies the V.I and also has*

$$\left\langle w, \nabla^2_{xx} L(\bar{x}, \bar{y})\, w \right\rangle \geq 0 \quad \forall\, w \in K.$$

(b) *(Sufficient Condition) If a multiplier vector $\bar{y}$ exists such that $(\bar{x}, \bar{y})$ satisfies the KKT conditions, or equivalently the V.I. and if the inequality in part (a) hold strictly when $w \neq 0$, then $\bar{x}$ furnishes a local minimum in the minimization problem. It is a strong local minimum in the sense that $\exists \epsilon > 0$ such that*

$$g_0(x) \geq g_0(\bar{x}) + \epsilon |x - \bar{x}|^2$$

Summary of this talk

Definitions and Preliminaries

Variational Inequalities with Monotonicity

- Convexity of solution
- Existence without monotonicity and boundedness (improving a 2A.1)
- Existence for monotone V.I.
- V.I. with strong monotonicity (property of the solutions)

Consequences for Optimization

- 2nd-Order optimality on polyhedral convex set
- Parametrized minimization problem
- Tilted minimization problem
- Minimization problem with system of constraints (not convex)

Thank you for attention!