# The Implicit Function and Inverse Function Theorems

Selina Klien

12. November 2015

# Outline

# Ideas Of Solving Equations

There are different ways to solve equations. Two ideas:

- Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. We want to solve the equation

  $$f(p, x) = 0.$$

  Idea: $x$ is a function of $p$ : $\quad x = s(p)$, such that

  $$f(p, s(p)) = 0.$$

  The function $x = s(p)$ is defined *implicitly* by the equation.

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. The idea of solving the equation

  $$f(x) = y$$

  for $x$ as a function of $y$ concerns the inversion of $f$.

# Important Theorems

When does such a function $s(p)$ or inversion of $f$ exist?

There are two well-known theorems, which guarantee us a at least **local** solution under special conditions:

- Implicit Function Theorem
- Inverse Function Theorem

# Outline

# Solution Mapping

'Problem': fix vector $p$ and we are looking for a 'solution' $x$ such that the equation $f(p,x) = 0$ holds.
$\Rightarrow$ solution mapping $S$ as set-valued mapping signaled by the notation:

$$S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$$
$$p \mapsto \{x \in \mathbb{R} | f(p,x) = 0\}$$

The graph of $S$ is the set

$$\text{gph} S = \{(p,x) \in \mathbb{R}^d \times \mathbb{R}^n | x \in S(p)\}$$

What are the properties of set-valued mappings?

# Properties of set-valued mappings

General set-valued mapping

$$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$$

with graph of $F$

$$\mathrm{gph}F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in F(x)\}$$

F is ...

... empty-valued at $x :\Leftrightarrow F(x) = \emptyset$

... single-valued at $x :\Leftrightarrow F(x) = y$ with $y \in \mathbb{R}^m$

... multivalued at $x :\Leftrightarrow F(x)$ assigns more than one element, $|F(x)| \geq 2$.

# Domain and Range of $F$

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. The domain of $F$ is the set

$$\text{dom}F = \{x | F(x) \neq \emptyset\}$$

while the range of $F$ is

$$\text{rge}F = \{y | y \in F(x) \textit{ for some } x\}$$

A *function* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ which ist single-valued at every point of $\text{dom}F$.

We can emphasize this by writing $F : \mathbb{R}^n \to \mathbb{R}^m$.

# Inverse of set-valued mappings

One advantage of the framework of set-valued mappings:

*Every set-valued mapping* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *has an inverse, namely the set valued mapping* $F^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *defined by*

$$F^{-1}(y) = \{x | y \in F(x)\}$$

In this manner a function $f$ **always** has an inverse $f^{-1}$ as a **set-valued mapping**.

When is the set-valued mapping $f^{-1}$ a function?

# Graphical Localization

For $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a pair $(\bar{x}, \bar{y}) \in \text{gph } F$, a graphical localization of $F$ at $\bar{x}$ for $\bar{y}$ is a set-valued mapping $\bar{F}$, such that

$$\text{gph}\bar{F} = (U \times V) \cap \text{ghp}F \textit{ for some neighborhoods } U \textit{ of } \bar{x} \textit{ and } V \textit{ of } \bar{y}$$

so that

$$\bar{F} : x \mapsto \begin{cases} F(x) \cap V & x \in U \\ \emptyset & \end{cases}$$

$$\bar{F}^{-1} : x \mapsto \begin{cases} F^{-1}(y) \cap U & y \in V \\ \emptyset & \end{cases}$$

# Single-Valued Localization

### Definition

By a single-valued localization of $F$ at $\bar{x}$ will be meant a graphical localization that is a function, its domain not necessarily being a neighbourhood of $\bar{x}$.
The case where the domain is indeed a neighbourhood of $\bar{x}$ will be indicated by referring to a single-valued localization of $F$ around $\bar{x}$ for $\bar{y}$, instead of just $\bar{x}$ for $\bar{y}$.

# The Solution Mapping

The solution mapping of $f(p, x) = 0$ is a set-valued mapping, which is defined by

$$S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$$
$$S(p) = \{x | f(p, x) = 0\}$$

We can look at pairs $(\bar{p}, \bar{x})$ in gph$S$ and ask whether $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$.
Such a localization is exactly what constitutes an implicit function coming out of the equation.

# Outline

# Classical Implicit Function Theorem

**Theorem**

*Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a neighbourhood of a point $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$, and let the partial Jacobian of $f$ with respect to $x$ at $(\bar{p}, \bar{x})$, namely $\nabla_x f(\bar{p}, \bar{x})$, be nonsingular.*
*Then the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable in a neighbourhood $Q$ of $\bar{p}$ with Jacobian satisfying*

$$\nabla s(p) = -\nabla_x f(p, s(p))^{-1} \nabla_p f(p, s(p)) \qquad \forall p \in Q$$

# Contraction Mapping Principle

**Theorem ( Contraction Mapping Principle)**

*Let $X$ be a complete metric space with metric $d$. Consider a point $\bar{x} \in X$ and function $\phi : X \to X$ for which there exist scalars $a > 0$ and $\lambda \in [0, 1)$ such that*

$$1. d(\phi(\bar{x}), \bar{x}) \leq a(1 - \lambda)$$
$$2. d(\phi(x'), \phi(x)) \leq \lambda\, d(x', x) \qquad \forall x, x' \in \mathbb{B}_a(\bar{x})$$

*Then there is a unique $x \in \mathbb{B}_a(\bar{x})$ satisfying $x = \phi(x)$.*

Proof: Analysis

Another equivalent version of contraction mapping principle:

# Parametric Contraction Mapping Principle

## Theorem

*Let $X$ be a complete metric space and , $P$ be a metric space with metrics $d_x$,$d_p$ and let $\phi : P \times X \to X$. Suppose that there exist a $\lambda \in [0,1)$ and $\mu \geq 0$ such that*

$$d_x(\phi(p,x'),\phi(p,x)) \leq \lambda d_x(x',x) \qquad \forall x,x' \in X \forall p \in P$$
$$d_x(\phi(p',x),\phi(p,x)) \leq \mu d_p(p',p) \qquad \forall p,p' \in P \forall x \in X$$

*then the mapping*

$$\Psi : p \mapsto \{x \in X | x = \phi(p,x)\} \qquad \text{for } p \in P$$

*is single-valued on $P$, which is moreover Lipschitz continuous on $P$ with Lipschitz constant $\mu/(1-\lambda)$.*

# Sketch Of Proof

**Step 1**: Existence of a single-valued localization $s(p) = x$
To show: The function

$$\psi : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(p, x) \mapsto x - D^{-1} f(p, x)$$

where $D := \nabla_x f(\bar{p}, \bar{x})$, satisfies the condition of the Parametric Contraction Mapping Principle - Theorem.

$\Rightarrow$ single-valued localization in a neighbourhood of $(\bar{p}, \bar{x})$.

$$s : p \mapsto \{x | x = \psi(p, x) = x - D^{-1} f(p, x)\}$$

**Step 2**: Derivative of $s$
Use Chainrule. Let $p \in U(\bar{p})$

$$\begin{aligned} 0 &= f(p, s(p)) \\ 0 &= \nabla_p f(p, s(p)) + \nabla_x f(p, s(p)) \nabla_p s(p) \\ \Leftrightarrow \nabla s(p) &= -\nabla_x f(p, s(p))^{-1} \nabla_p f(p, s(p)) \end{aligned}$$

**Step 3**: Continuous differentiability of $s$

$f$ is continuously differentiable and $\nabla_x f(\bar{p}, \bar{x})$ is nonsingular.

# Outline

# Classical Inverse Function Theorem

**Theorem**

*Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a neighbourhood of a point $\bar{x}$ and let $\bar{y} = f(\bar{x})$.*
*If $\nabla f(\bar{x})$ is nonsingular, then $f^{-1}$ has a single-valued localization $s$ around $\bar{y}$ for $\bar{x}$. Moreover, the function $s$ is continuously differentiable in a neighbourhood $V$ of $\bar{y}$, and its Jacobian satisfies*

$$\nabla s(y) = \nabla f(s(y))^{-1} \qquad \forall y \in V$$

# Sketch Of Proof

The Inverse Function Theorem is a special case of the Implicit Function Theorem:

Sketch of Proof: Let $\bar{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\bar{f}(x,y) = f(x) - y$. $f$ is continuously differentiable and hence $\bar{f}$ is continuously differentiable and $\nabla_x \bar{f}(\bar{x},\bar{y}) = \nabla f(\bar{x})$ is nonsingular.

$\Rightarrow$ a single valued function exists:

$$s : y \mapsto \{x | f(x) - y = 0\} = \{x | f(x) = y\}$$

And

$$\nabla s(y) = -\nabla_x \bar{f}(x,y)^{-1} \nabla_y \bar{f}(x,y) = \nabla f(x)^{-1}$$

Are there any conditions such that the following two statements are equivalent?

- the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable in a neighbourhood $Q$ of $\bar{p}$
- $\nabla_x f(\bar{p}, \bar{x})$ is nonsingular.

**Answer:** Yes! The condition: $\nabla_p f(\bar{p}, \bar{x})$ has full rank $n$.

# Outline

# Symmetric Implicit Function Theorem

**Theorem (Symmetric Implicit Function Theorem)**

*Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a neighbourhood of $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$, and let $\nabla_p f(\bar{p}, \bar{x})$ be of full rank $n$. Then the following are equivalent*

- *the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable in a neighbourhood $Q$ of $\bar{p}$*
- *$\nabla_x f(\bar{p}, \bar{x})$ is nonsingular.*

# Symmetric Inverse Function Theorem

**Theorem (Symmetric Inverse Function Theorem)**

*Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable around $\bar{x}$. Then the following are equivalent*

- $\nabla f(\bar{x})$ *is nonsingular.*
- $f^{-1}$ *has a single-valued localization around $\bar{y} := f(\bar{x})$ for $\bar{x}$ which is continuously differentiable around $\bar{y}$.*

Proof: Classical Inverse Function Theorem + Symmetric Implicit Function Theorem

# Outline

# Calmness

### Definition (Calmness)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be calm at $\bar{x}$ relative to a set $D$ in $\mathbb{R}^n$ if $\bar{x} \in D \cap \operatorname{dom} f$ and there exists a constant $\kappa \geq 0$ such that

$$|f(x) - f(\bar{x})| \leq \kappa |x - \bar{x}| \qquad \forall x \in D \cap \operatorname{dom} f$$

### Definition (Lipschitz Continuous Functions)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz continuous relative to a set $D$, or on a set $D$, if $D \subset \operatorname{dom} f$ and there exists a constant $\kappa \geq 0$ such that

$$|f(x) - f(x')| \leq \kappa |x - x'| \qquad \forall x, x' \in D$$

# Calmness Modulus

### Definition (Calmness Modulus)

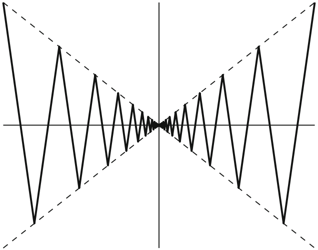
The calmness modulus of $f$ at $\bar{x}$, denoted $\mathrm{clm}(f;\bar{x})$ is defined by

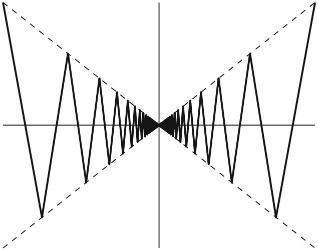
$$\mathrm{clm}(f;\bar{x}) := \limsup_{x \in \mathrm{dom} f, x \to \bar{x}, x \neq \bar{x}} \frac{|f(x) - f(\bar{x})|}{|x - \bar{x}|}$$

It is obvious that

$$f \textit{ is calm at } \bar{x} \iff \mathrm{clm}(f;\bar{x}) < \infty$$

# Lipschitz vs Calmness





$$1. f(x) = (-1)^{n+1} 9x + (-1)^n \frac{2^{2n+1}}{5^{n-2}},\ |x| \in \left[\frac{4^n}{5^{n-1}}, \frac{4^{n-1}}{5^{n-2}}\right]$$

$$2.\ f(x) = (-1)^{n+1}(6+n)x + (-1)^n 210 \frac{(5+n)!}{(6+n)!},$$

$$|x| \in \left[210\frac{(5+n)!}{(7+n)!}, 210\frac{(4+n)!}{(6+n)!}\right]$$

# Properties of Calmness Modulus

- $\text{clm}(f;\bar{x}) \geq 0$ for every $\bar{x} \in \text{dom}f$
- $\text{clm}(\lambda f;\bar{x}) = |\lambda|\text{clm}(f;\bar{x})$ for any $\lambda \in \mathbb{R}$ and $\bar{x} \in \text{dom}f$
- $\text{clm}(f+g;\bar{x}) \leq \text{clm}(f;\bar{x}) + \text{clm}(g;\bar{x})$ any $\forall \bar{x} \in \text{dom}f \cap \text{dom}g$

- $\text{clm}(f \circ g;\bar{x}) \leq \text{clm}(f;\bar{x}) \cdot \text{clm}(g;\bar{x})$ whenever $\bar{x} \in \text{dom}g$ and $g(\bar{x}) \in \text{dom}\ f$
- $\text{clm}(f+g;\bar{x}) = 0 \Rightarrow \text{clm}(f;\bar{x}) = \text{clm}(g;\bar{x})$ whenever $\bar{x} \in \text{dom}g \cap \text{dom}f$
  (converse is false!!!)

# Partial Calmness

## Definition

A function $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be calm w. r. t. $x$ at $(\bar{p}, \bar{x}) \in \operatorname{dom} f$ when the function $\phi$ with values $\phi(\bar{x}) = f(\bar{p}, x)$ is calm at $\bar{x}$.
Such calmness is said to be uniform in $p$ at $(\bar{p}, \bar{x})$ when there exists a constant $\kappa \geq 0$ and neighbourhoods $Q$ of $p$ and $U$ of $\bar{x}$ such that actually

$$|f(p,x) - f(p,\bar{x})| \leq \kappa |x - \bar{x}| \qquad \forall (p,x) \in (Q \times U) \cap \operatorname{dom} f$$

The partial calmness modulus of $f$ w. r. t. $x$ at $(\bar{p}, \bar{x})$ is denoted as $\operatorname{clm}_x(f; (\bar{p}, \bar{x}))$.
While the uniform partial calmness modulus is

$$\widehat{\operatorname{clm}}_x(f; (\bar{p}, \bar{x})) := \limsup_{x \to \bar{x}, p \to \bar{p}, (p,x) \in \operatorname{dom} f, x \neq \bar{x}} \frac{|f(p,x) - f(p,\bar{x})|}{|x - \bar{x}|}.$$

The Theorems shows that the invertibility of the derivative is a necessary condition to obtain a calm single valued localization of the inverse.

**Theorem (Jacobian Nonsingularity from Inverse Calmness)**

*Given $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\bar{x} \in int\, dom f$. Let $f$ be differentiable at $\bar{x}$ and let $\bar{y} = f(\bar{x})$.*
*If $f^{-1}$ has a single-valued localization around $\bar{y}$ for $\bar{x}$, which is calm at $\bar{y}$ then $\nabla f(\bar{x})$ is nonsingular.*

# Outline

# Lipschitz Continuity

### Definition (Lipschitz Continuous Functions)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Lipschitz continuous relative to a set $D$, or on a set $D$, if $D \subset \mathrm{dom} f$ and there exists a constant $\kappa \geq 0$ such that

$$|f(x) - f(x')| \leq \kappa |x - x'| \qquad \forall x, x' \in D$$

### Definition (Lipschitz Modulus)

The Lipschitz modulus of $f$ at $\bar{x}$, denoted $\mathrm{clm}(f; \bar{x})$ is defined by

$$\mathrm{lip}(f; \bar{x}) := \limsup_{x', x \to \bar{x}, x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}$$

It is obvious that

$$f \textit{ is Lipschitz continuous around } \bar{x} \iff \mathrm{lip}(f; \bar{x}) < \infty$$

# Lipschitz Continuity from Differentiability

**Theorem**

*If $f$ is continuously differentiable on an open set $O$ and $C$ is a compact convex subset of $O$, then $f$ is Lipschitz continuous relative to $C$ with constant*

$$\kappa = \max_{x \in C} |\nabla f(x)|.$$

Proof: Analysis.

Easy Examples:

- The function $x \mapsto x^2$ is differentiable on $\mathbb{R}$, but not Lipschitz continuous on $\mathbb{R}$.
- The function $x \mapsto |x|$ is Lipschitz continuous with $\text{lip}(|x|, x) = 1$ on $\mathbb{R}$ but it is not differentiable at 0.

# Strict Differentiability

## Definition (Strict Differentiability)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be strictly differentiable at point $\bar{x}$ if there is a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\operatorname{lip}(e;\bar{x}) = 0 \text{ for } e(x) = f(x) - [f(\bar{x}) + A(x-\bar{x})]$$

$$\operatorname{lip}(e;\bar{x}) = \limsup_{x',x\to\bar{x},x\neq x'} \frac{|f(x) - [f(x') + A(x-x')]|}{|x-x'|}$$

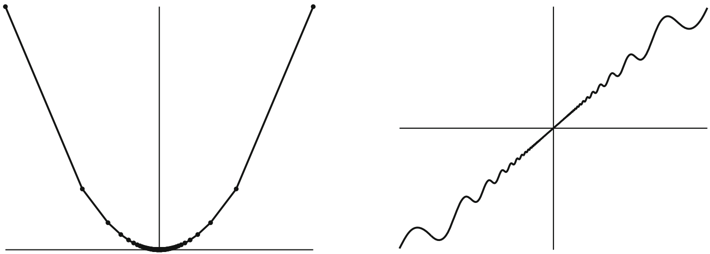
## Definition

A function f is differentiable if there exists a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\operatorname{clm}(e,\bar{x}) = 0, \qquad \operatorname{clm}(e,\bar{x}) = \limsup_{x\in\operatorname{dom}f,x\to\bar{x},x\neq\bar{x}} \frac{|f(x) - [f(\bar{x}) + A(x-\bar{x})]|}{|x-\bar{x}|}$$

If $f$ is differentiable at $\bar{x}$ then $A = \nabla f(\bar{x})$.

# Strict Differentiability vs Differentiability



1. $f(0) = 0,\ f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ *and it is linear in the Intervals* $\left[\frac{1}{n}, \frac{1}{n+1}\right]$
2. $f(0) = 0,\ f(x) = \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right)$

# Properties of Strict Differentiability

- $f$ is continuously differentiable in a neighbourhood of $\bar{x} \Rightarrow f$ is strictly differentiable at $\bar{x}$
- $f$ is strictly differentiable on an open set $O \iff f$ is continuously differentiable on $O$
- $f$ is differentiable at every point in a neighbourhood of $\bar{x}$. Then

$$f \textit{ is strictly differentiable at } \bar{x} \iff \nabla f \textit{ ist continuous at } \bar{x}$$

# Outline

# Symmetric Inverse Function Theorem Under Strict Differentiability

**Theorem (Symmetric Inverse Function Theorem Under Strict Differentiability)**

*Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly differentiable at $\bar{x}$. Then the following are equivalent*

- $\nabla f(\bar{x})$ *is nonsingular*
- $f^{-1}$ *has a single-valued localization $s$ around $\bar{y} := f(\bar{x})$ for $\bar{x}$ which is strictly differentiable at $\bar{y}$. In that case, moreover*

$$\nabla s(\bar{y}) = \nabla f(\bar{x})^{-1}$$

# Strict Partial Differentiability

## Definition (Strict Partial Differentiability)

A function $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ is said to be strictly differentiable with respect to $x$ at $(\bar{p}, \bar{x})$ if the function $x \mapsto f(\bar{p}, x)$ is strictly differentiable at $\bar{x}$. It is said to be strictly differentiable with respect to x uniformly in $p$ at $(\bar{p}, \bar{x})$ if for every $\epsilon > 0$ there are neighbourhoods $Q$ of $\bar{p}$ and $U$ of $\bar{x}$ such that

$$|f(p,x) - [f(p,x') + D_x f(\bar{p},\bar{x})(x - x')]| \leq \epsilon |x - x'| \qquad \forall x, x' \in U \forall p \in Q.$$

# Implicit Functions Under Strict Partial Differentiability

**Theorem (Implicit Functions Under Strict Partial Differentiability)**

*Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be strictly differentiable at a point $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$ and let the partial Jacobian $\nabla_x f(\bar{p}, \bar{x})$ be nonsingular. Then the solution mapping*

$$S : p \mapsto \{x \in \mathbb{R}^n | f(p, x) = 0\}$$

*has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is strictly differentiable at $\bar{p}$ with its Jacobian expressed by*

$$\nabla s(\bar{p}) = -\nabla_x f(\bar{p}, \bar{x})^{-1} \nabla_p f(\bar{p}, \bar{x}).$$

Proof: Similar to the proof of the Classic Implicit Function Theorem.

# Summary - Inverse Function Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable around / strictly differentiable at $\bar{x}$ and let $\bar{y} = f(\bar{x})$. Then the following are equivalent

- $\nabla f$ is nonsingular
- $f^{-1}$ has a single-valued localization $s$ around $\bar{y} := f(\bar{x})$ for $\bar{x}$ which is continuously differentiable around / strictly differentiable at $\bar{y}$. In that case, moreover

$$\nabla s(y) = \nabla f(s(y))^{-}1 \qquad y \in U_\epsilon(\bar{y})$$
$$\nabla s(\bar{y}) = \nabla f(\bar{x})^{-1}$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at $\bar{x} \in \operatorname{int} \operatorname{dom} f$ and let $\bar{y} = f(\bar{x})$. If $f^{-1}$ has a single-valued localization around $\bar{y}$ for $\bar{x}$, which is calm at $\bar{y}$ then $\nabla f(\bar{x})$ is nonsingular.

# Summary - Implicit Functions

Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable around / strictly differentiable at $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$ and let the partial Jacobian $\nabla_x f(\bar{p}, \bar{x})$ be nonsingular.

Then the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable around / strictly differentiable at $\bar{p}$ with its Jacobian expressed by

$$\nabla s(p) = -\nabla_x f(p, s(p))^{-1} \nabla_p f(p, s(p)) \qquad p \in U_\epsilon(\bar{p})$$

$$\nabla s(\bar{p}) = -\nabla_x f(\bar{p}, \bar{x})^{-1} \nabla_p f(\bar{p}, \bar{x}).$$

# Thank you for your attention