The Implicit Function and Inverse Function Theorems

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Outline

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   - Important Definitions

2. Implicit Function Theorem and Inverse Function Theorem
   - Classical Implicit Function Theorem
   - Classical Inverse Function Theorem
   - Symmetric Function Theorems

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   - Definition

4. Lipschitz Continuity
   - Definition
   - Symmetric Inverse/Implicit Function Theorem Under Strict Differentiability
Ideas Of Solving Equations

There are different ways to solve equations. Two ideas:

- Let $f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. We want to solve the equation
  \[ f(p, x) = 0. \]
  Idea: $x$ is a function of $p$: $x = s(p)$, such that
  \[ f(p, s(p)) = 0. \]
  The function $x = s(p)$ is defined *implicitly* by the equation.

- Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. The idea of solving the equation
  \[ f(x) = y \]
  for $x$ as a function of $y$ concerns the inversion of $f$. 

Important Theorems

When does such a function \( s(p) \) or inversion of \( f \) exist?

There are two well-known theorems, which guarantee us a at least local solution under special conditions:

- Implicit Function Theorem
- Inverse Function Theorem
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Solution Mapping

'Problem': fix vector $p$ and we are looking for a 'solution' $x$ such that the equation $f(p, x) = 0$ holds.

⇒ solution mapping $S$ as set-valued mapping signaled by the notation:

$$S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$$

$$p \mapsto \{x \in \mathbb{R} | f(p, x) = 0\}$$

The graph of $S$ is the set

$$\text{gph}S = \{(p, x) \in \mathbb{R}^d \times \mathbb{R}^n | x \in S(p)\}$$

What are the properties of set-valued mappings?
Properties of set-valued mappings

General set-valued mapping

\[ F : \mathbb{R}^n \Rightarrow \mathbb{R}^m \]

with graph of \( F \)

\[ \text{gph} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in F(x)\} \]

\( F \) is …

… empty-valued at \( x \) :\(\iff F(x) = \emptyset \)

… single-valued at \( x \) :\(\iff F(x) = y \ with \ y \in \mathbb{R}^m \)

… multivalued at \( x \) :\(\iff F(x) \) assigns more than one element, \(|F(x)| \geq 2.\)
Domain and Range of $F$

Let $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ be a set-valued mapping. The domain of $F$ is the set

$$\text{dom}F = \{ x \mid F(x) \neq \emptyset \}$$

while the range of $F$ is

$$\text{rge}F = \{ y \mid y \in F(x) \text{ for some } x \}$$

A function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a set-valued mapping $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ which is single-valued at every point of $\text{dom}F$.

We can emphasize this by writing $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$. 
Inverse of set-valued mappings

One advantage of the framework of set-valued mappings:

Every set-valued mapping $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ has an inverse, namely the set valued mapping $F^{-1} : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ defined by

$$F^{-1}(y) = \{x|y \in F(x)\}$$

In this manner a function $f$ **always** has an inverse $f^{-1}$ as a **set-valued mapping**.

When is the set-valued mapping $f^{-1}$ a function?
For $F : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ and a pair $(\bar{x}, \bar{y}) \in \text{gph} \ F$, a graphical localization of $F$ at $\bar{x}$ for $\bar{y}$ is a set-valued mapping $\bar{F}$, such that

$\text{gph} \bar{F} = (U \times V) \cap \text{gph} F$ for some neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ so that

$$\bar{F} : x \mapsto \begin{cases} F(x) \cap V & x \in U \\ \emptyset & \end{cases}$$

$$\bar{F}^{-1} : x \mapsto \begin{cases} F^{-1}(y) \cap U & y \in V \\ \emptyset & \end{cases}$$
Single-Valued Localization

Definition

By a single-valued localization of $F$ at $\bar{x}$ will be meant a graphical localization that is a function, its domain not necessarily being a neighbourhood of $\bar{x}$.

The case where the domain is indeed a neighbourhood of $\bar{x}$ will be indicated by referring to a single-valued localization of $F$ around $\bar{x}$ for $\bar{y}$, instead of just $\bar{x}$ for $\bar{y}$.
The solution mapping of \( f(p, x) = 0 \) is a set-valued mapping, which is defined by

\[
S : \mathbb{R}^d \Rightarrow \mathbb{R}^n
\]

\[
S(p) = \{ x | f(p, x) = 0 \}
\]

We can look at pairs \((\bar{p}, \bar{x})\) in \( \text{gph}S \) and ask whether \( S \) has a single-valued localization \( s \) around \( \bar{p} \) for \( \bar{x} \).

Such a localization is exactly what constitutes an implicit function coming out of the equation.
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**Classical Implicit Function Theorem**

**Theorem**

Let \( f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n \) be continuously differentiable in a neighbourhood of a point \((\bar{p}, \bar{x})\) and such that \( f(\bar{p}, \bar{x}) = 0\), and let the partial Jacobian of \( f \) with respect to \( x \) at \((\bar{p}, \bar{x})\), namely \( \nabla_x f(\bar{p}, \bar{x}) \), be nonsingular. Then the solution mapping \( S \) has a single-valued localization \( s \) around \( \bar{p} \) for \( \bar{x} \) which is continuously differentiable in a neighbourhood \( Q \) of \( \bar{p} \) with Jacobian satisfying

\[
\nabla s(p) = -\nabla_x f(p, s(p))^{-1} \nabla_p f(p, s(p)) \quad \forall p \in Q
\]
Theorem (Contraction Mapping Principle)

Let $X$ be a complete metric space with metric $d$. Consider a point $\bar{x} \in X$ and function $\phi : X \to X$ for which there exist scalars $a > 0$ and $\lambda \in [0, 1)$ such that

1. $d(\phi(\bar{x}), \bar{x}) \leq a(1 - \lambda)$
2. $d(\phi(x'), \phi(x)) \leq \lambda d(x', x)$ \hspace{1cm} \forall x, x' \in B_a(\bar{x})$

Then there is a unique $x \in B_a(\bar{x})$ satisfying $x = \phi(x)$.

Proof: Analysis

Another equivalent version of contraction mapping principle:
Parametric Contraction Mapping Principle

Theorem

Let $X$ be a complete metric space and $P$ be a metric space with metrics $d_x, d_p$ and let $\phi : P \times X \to X$. Suppose that there exist a $\lambda \in [0, 1)$ and $\mu \geq 0$ such that

$$
\begin{align*}
d_x(\phi(p, x'), \phi(p, x)) &\leq \lambda d_x(x', x) & \forall x, x' \in X \forall p \in P \\
d_x(\phi(p', x), \phi(p, x)) &\leq \mu d_p(p', p) & \forall p, p' \in P \forall x \in X
\end{align*}
$$

then the mapping

$$
\Psi : p \mapsto \{x \in X| x = \phi(p, x)\}
$$

is single-valued on $P$, which is moreover Lipschitz continuous on $P$ with Lipschitz constant $\mu/(1 - \lambda)$. 
Sketch Of Proof

**Step 1**: Existence of a single-valued localization $s(p) = x$

To show: The function

$$
\psi : \mathbb{R}^q \times \mathbb{R}^n \to \mathbb{R}^n
$$

$$(p, x) \mapsto x - D^{-1}f(p, x)$$

where $D := \nabla_x f(\bar{p}, \bar{x})$, satisfies the condition of the Parametric Contraction Mapping Principle - Theorem.

$\Rightarrow$ single-valued localization in a neighbourhood of $(\bar{p}, \bar{x})$.

$$s : p \mapsto \{x | x = \psi(p, x) = x - D^{-1}f(p, x)\}$$
**Step 2:** Derivative of $s$

Use Chainrule. Let $p \in U(\bar{p})$

$$0 = f(p, s(p))$$
$$0 = \nabla_p f(p, s(p)) + \nabla_x f(p, s(p))\nabla_p s(p)$$
$$\iff \nabla s(p) = -\nabla_x f(p, s(p))^{-1}\nabla_p f(p, s(p))$$

**Step 3:** Continuous differentiability of $s$

$f$ is continuously differentiable and $\nabla_x f(\bar{p}, \bar{x})$ is nonsingular.
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Classical Inverse Function Theorem

**Theorem**

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable in a neighbourhood of a point $\bar{x}$ and let $\bar{y} = f(\bar{x})$. If $\nabla f(\bar{x})$ is nonsingular, then $f^{-1}$ has a single-valued localization $s$ around $\bar{y}$ for $\bar{x}$. Moreover, the function $s$ is continuously differentiable in a neighbourhood $V$ of $\bar{y}$, and its Jacobian satisfies

$$\nabla s(y) = \nabla f(s(y))^{-1} \quad \forall y \in V$$
Sketch Of Proof

The Inverse Function Theorem is a special case of the Implicit Function Theorem:

Sketch of Proof: Let \( \tilde{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) be defined by \( \tilde{f}(x, y) = f(x) - y \). \( f \) is continuously differentiable and hence \( \tilde{f} \) is continuously differentiable and \( \nabla_x \tilde{f}(\bar{x}, \bar{y}) = \nabla f(\bar{x}) \) is nonsingular.

\( \Rightarrow \) a single valued function exists:

\[
    s : y \mapsto \{ x | f(x) - y = 0 \} = \{ x | f(x) = y \}
\]

And

\[
    \nabla s(y) = -\nabla_x \tilde{f}(x, y)^{-1} \nabla_y \tilde{f}(x, y) = \nabla f(x)^{-1}
\]
Are there any conditions such that the following two statements are equivalent?

- the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable in a neighbourhood $Q$ of $\bar{p}$
- $\nabla_x f(\bar{p}, \bar{x})$ is nonsingular.

**Answer:** Yes! The condition: $\nabla_p f(\bar{p}, \bar{x})$ has full rank $n$. 
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Symmetric Implicit Function Theorem

Theorem (Symmetric Implicit Function Theorem)

Let \( f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuously differentiable in a neighbourhood of \((\bar{p}, \bar{x})\) and such that \( f(\bar{p}, \bar{x}) = 0\), and let \( \nabla_p f(\bar{p}, \bar{x}) \) be of full rank \( n \). Then the following are equivalent

- the solution mapping \( S \) has a single-valued localization \( s \) around \( \bar{p} \) for \( \bar{x} \) which is continuously differentiable in a neighbourhood \( Q \) of \( \bar{p} \)
- \( \nabla_x f(\bar{p}, \bar{x}) \) is nonsingular.
Symmetric Inverse Function Theorem

**Theorem (Symmetric Inverse Function Theorem)**

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable around $\bar{x}$. Then the following are equivalent

- $\nabla f(\bar{x})$ is nonsingular.
- $f^{-1}$ has a single-valued localization around $\bar{y} := f(\bar{x})$ for $\bar{x}$ which is continuously differentiable around $\bar{y}$.

**Proof:** Classical Inverse Function Theorem + Symmetric Implicit Function Theorem
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Calmness

**Definition (Calmness)**

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be calm at \( \bar{x} \) relative to a set \( D \) in \( \mathbb{R}^n \) if \( \bar{x} \in D \cap \text{dom}f \) and there exists a constant \( \kappa \geq 0 \) such that

\[
|f(x) - f(\bar{x})| \leq \kappa |x - \bar{x}| \quad \forall x \in D \cap \text{dom}f
\]

**Definition (Lipschitz Continuous Functions)**

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is said to be Lipschitz continuous relative to a set \( D \), or on a set \( D \), if \( D \subset \text{dom}f \) and there exists a constant \( \kappa \geq 0 \) such that

\[
|f(x) - f(x')| \leq \kappa |x - x'| \quad \forall x, x' \in D
\]
Definition (Calmness Modulus)

The calmness modulus of $f$ at $\bar{x}$, denoted $\text{clm}(f; \bar{x})$ is defined by

$$\text{clm}(f; \bar{x}) := \limsup_{x \in \text{dom} f, x \to \bar{x}, x \neq \bar{x}} \frac{|f(x) - f(\bar{x})|}{|x - \bar{x}|}$$

It is obvious that

$f$ is calm at $\bar{x} \iff \text{clm}(f; \bar{x}) < \infty$
Lipschitz vs Calmness

1. \( f(x) = (-1)^{n+1}9x + (-1)^n \frac{2^{2n+1}}{5^{n-2}}, \quad |x| \in \left[ \frac{4^n}{5^{n-1}}, \frac{4^{n-1}}{5^{n-2}} \right] \)

2. \( f(x) = (-1)^{n+1}(6 + n)x + (-1)^n 210 \frac{(5 + n)!}{(6 + n)!}, \quad |x| \in \left[ \frac{210(5 + n)!}{(7 + n)!}, \frac{210(4 + n)!}{(6 + n)!} \right] \)
Properties of Calmness Modulus

- $\text{clm}(f; \bar{x}) \geq 0$ for every $\bar{x} \in \text{dom} f$
- $\text{clm}(\lambda f; \bar{x}) = |\lambda| \text{clm}(f; \bar{x})$ for any $\lambda \in \mathbb{R}$ and $\bar{x} \in \text{dom} f$
- $\text{clm}(f + g; \bar{x}) \leq \text{clm}(f; \bar{x}) + \text{clm}(g; \bar{x})$ any $\forall \bar{x} \in \text{dom} f \cap \text{dom} g$
- $\text{clm}(f \circ g; \bar{x}) \leq \text{clm}(f; \bar{x}) \cdot \text{clm}(g; \bar{x})$ whenever $\bar{x} \in \text{dom} g$ and $g(\bar{x}) \in \text{dom} f$
- $\text{clm}(f + g; \bar{x}) = 0 \implies \text{clm}(f; \bar{x}) = \text{clm}(g; \bar{x})$ whenever $\bar{x} \in \text{dom} g \cap \text{dom} f$
  (converse is false!!!)
Partial Calmness

Definition

A function $f : \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be calm w. r. t. $x$ at $(\bar{p}, \bar{x}) \in \text{dom} f$ when the function $\phi$ with values $\phi(\bar{x}) = f(\bar{p}, x)$ is calm at $\bar{x}$. Such calmness is said to be uniform in $p$ at $(\bar{p}, \bar{x})$ when there exists a constant $\kappa \geq 0$ and neighbourhoods $Q$ of $p$ and $U$ of $\bar{x}$ such that actually

$$|f(p, x) - f(p, \bar{x})| \leq \kappa |x - \bar{x}| \quad \forall (p, x) \in (Q \times U) \cap \text{dom} f$$

The partial calmness modulus of $f$ w. r. t. $x$ at $(\bar{p}, \bar{x})$ is denoted as $\text{clm}_x(f; (\bar{p}, \bar{x}))$.

While the uniform partial calmness modulus is

$$\hat{\text{clm}}_x(f; (\bar{p}, \bar{x})) := \limsup_{x \rightarrow \bar{x}, p \rightarrow \bar{p}, (p, x) \in \text{dom} f, x \neq \bar{x}} \frac{|f(p, x) - f(p, \bar{x})|}{|x - \bar{x}|}.$$
The Theorems shows that the invertibility of the derivative is a necessary condition to obtain a calm single valued localization of the inverse.

**Theorem (Jacobian Nonsingularity from Inverse Calmness)**

*Given* \( f : \mathbb{R}^n \to \mathbb{R}^n \) *and* \( \bar{x} \in \text{int dom} f \). *Let* \( f \) *be differentiable at* \( \bar{x} \) *and let* \( \bar{y} = f(\bar{x}) \).

*If* \( f^{-1} \) *has a single-valued localization around* \( \bar{y} \) *for* \( \bar{x} \), *which is calm at* \( \bar{y} \) *then* \( \nabla f(\bar{x}) \) *is nonsingular.*
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Lipschitz Continuity

Definition (Lipschitz Continuous Functions)
A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be Lipschitz continuous relative to a set $D$, or on a set $D$, if $D \subset \text{dom}f$ and there exists a constant $\kappa \geq 0$ such that

$$|f(x) - f(x')| \leq \kappa |x - x'| \quad \forall x, x' \in D$$

Definition (Lipschitz Modulus)
The Lipschitz modulus of $f$ at $\bar{x}$, denoted $clm(f; \bar{x})$ is defined by

$$\text{lip}(f; \bar{x}) := \limsup_{x', x \to \bar{x}, x \neq x'} \frac{|f(x) - f(x')|}{|x - x'|}$$

It is obvious that

$$f \text{ is Lipschitz continuous around } \bar{x} \iff \text{lip}(f; \bar{x}) < \infty$$
Lipschitz Continuity from Differentiability

Theorem

If $f$ is continuously differentiable on an open set $O$ and $C$ is a compact convex subset of $O$, then $f$ is Lipschitz continuous relative to $C$ with constant

$$\kappa = \max_{x \in C} |\nabla f(x)|.$$

Proof: Analysis.

Easy Examples:

- The function $x \mapsto x^2$ is differentiable on $\mathbb{R}$, but not Lipschitz continuous on $\mathbb{R}$.
- The function $x \mapsto |x|$ is Lipschitz continuous with $\text{lip}(|x|, x) = 1$ on $\mathbb{R}$ but it is not differentiable at 0.
Strict Differentiability

Definition (Strict Differentiability)
A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be strictly differentiable at point $\bar{x}$ if there is a linear mapping $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\text{lip}(e; \bar{x}) = 0 \text{ for } e(x) = f(x) - [f(\bar{x}) + A(x - \bar{x})]$$

$$\text{lip}(e; \bar{x}) = \limsup_{x', x \to \bar{x}, x \neq x'} \left| \frac{f(x) - [f(x') + A(x - x')]}{|x - x'|} \right|$$

Definition
A function $f$ is differentiable if there exists a linear mapping $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\text{clm}(e, \bar{x}) = 0, \quad \text{clm}(e, \bar{x}) = \limsup_{x \in \text{dom } f, x \to \bar{x}, x \neq \bar{x}} \left| \frac{f(x) - [f(\bar{x}) + A(x - \bar{x})]}{|x - \bar{x}|} \right|$$

If $f$ is differentiable at $\bar{x}$ then $A = \nabla f(\bar{x})$. 
Strict Differentiability vs Differentiability

1. $f(0) = 0$, $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ and it is linear in the intervals $\left[\frac{1}{n}, \frac{1}{n+1}\right]$

2. $f(0) = 0$, $f(x) = \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right)$
Properties of Strict Differentiability

- $f$ is continuously differentiable in a neighbourhood of $\bar{x} \Rightarrow f$ is strictly differentiable at $\bar{x}$
- $f$ is strictly differentiable on an open set $O \iff f$ is continuously differentiable on $O$
- $f$ is differentiable at every point in a neighbourhood of $\bar{x}$. Then

\[ f \text{ is strictly differentiable at } \bar{x} \iff \nabla f \text{ is continuous at } \bar{x} \]
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Symmetric Inverse Function Theorem Under Strict Differentiability

Theorem (Symmetric Inverse Function Theorem Under Strict Differentiability)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly differentiable at $\bar{x}$. Then the following are equivalent

1. $\nabla f(\bar{x})$ is nonsingular
2. $f^{-1}$ has a single-valued localization $s$ around $\bar{y} := f(\bar{x})$ for $\bar{x}$ which is strictly differentiable at $\bar{y}$. In that case, moreover

$$\nabla s(\bar{y}) = \nabla f(\bar{x})^{-1}$$
Strict Partial Differentiability

Definition (Strict Partial Differentiability)
A function $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ is said to be strictly differentiable with respect to $x$ at $(\bar{p}, \bar{x})$ if the function $x \mapsto f(\bar{p}, x)$ is strictly differentiable at $\bar{x}$. It is said to be strictly differentiable with respect to $x$ uniformly in $p$ at $(\bar{p}, \bar{x})$ if for every $\epsilon > 0$ there are neighbourhoods $Q$ of $\bar{p}$ and $U$ of $\bar{x}$ such that

$$|f(p, x) - [f(p, x') + D_x f(\bar{p}, \bar{x})(x - x')]| \leq \epsilon |x - x'| \quad \forall x, x' \in U \forall p \in Q.$$
Implicit Functions Under Strict Partial Differentiability

Theorem (Implicit Functions Under Strict Partial Differentiability)

Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be strictly differentiable at a point $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$ and let the partial Jacobian $\nabla_x f(\bar{p}, \bar{x})$ be nonsingular. Then the solution mapping

$$S : p \mapsto \{ x \in \mathbb{R}^n | f(p, x) = 0 \}$$

has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is strictly differentiable at $\bar{p}$ with its Jacobian expressed by

$$\nabla s(\bar{p}) = -\nabla_x f(\bar{p}, \bar{x})^{-1} \nabla_p f(\bar{p}, \bar{x}).$$

Proof: Similar to the proof of the Classic Implicit Function Theorem.
Summary - Inverse Function Theorem

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be continuously differentiable around / strictly differentiable at \( \bar{x} \) and let \( \bar{y} = f(\bar{x}) \). Then the following are equivalent

- \( \nabla f \) is nonsingular
- \( f^{-1} \) has a single-valued localization \( s \) around \( \bar{y} := f(\bar{x}) \) for \( \bar{x} \) which is continuously differentiable around / strictly differentiable at \( \bar{y} \). In that case, moreover

\[
\nabla s(y) = \nabla f(s(y))^{-1} \quad y \in U_\epsilon(\bar{y})
\]
\[
\nabla s(\bar{y}) = \nabla f(\bar{x})^{-1}
\]

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be differentiable at \( \bar{x} \in \text{int dom} f \) and let \( \bar{y} = f(\bar{x}) \). If \( f^{-1} \) has a single-valued localization around \( \bar{y} \) for \( \bar{x} \), which is calm at \( \bar{y} \) then \( \nabla f(\bar{x}) \) is nonsingular.
Summary - Implicit Functions

Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable around / strictly differentiable at $(\bar{p}, \bar{x})$ and such that $f(\bar{p}, \bar{x}) = 0$ and let the partial Jacobian $\nabla_x f(\bar{p}, \bar{x})$ be nonsingular.

Then the solution mapping $S$ has a single-valued localization $s$ around $\bar{p}$ for $\bar{x}$ which is continuously differentiable around / strictly differentiable at $\bar{p}$ with its Jacobian expressed by

$$\nabla s(p) = -\nabla_x f(p, s(p))^{-1} \nabla_p f(p, s(p)) \quad p \in U_\epsilon(\bar{p})$$

$$\nabla s(\bar{p}) = -\nabla_x f(\bar{p}, \bar{x})^{-1} \nabla_p f(\bar{p}, \bar{x}).$$
Thank you for your attention