Extended Robinson, parametrizations and semiderivatives

Talk nr.4

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Overview

1. Robinson's Implicit Function Theorem
   - Previous formulation
   - Special case
2. Parametrizations
   - Ampleness
   - Solution mappings
3. Semiderivatives
   - New Implicit Function Theorem
4. Piecewise Smooth Functions

Robinson Theorem Extended Beyond Differentiability

Theorem (2B.5 Robinson Implicit Function Theorem)

If

1. $f(.,\overline{x})$ *is continuous at* $\overline{p}$ *and* h *a strict* **estimator** *of* f *wrt.* x *unif. in* p *at* $(\overline{p}, \overline{x})$ *with constant* μ*;*
2. *the inverse* G^{-1} *of* $G = h + F, 0 \in G(\overline{x})$ *has a Lip-cont. svl.* σ *around 0 with* $lip(\sigma; 0) \leq \kappa$ *and* $\kappa\mu < 1$*.*

then $S(p) = \{x | f(p,x) + F(x) \ni 0\}$ *has a svl.* s *around* $\overline{p}$ *for* $\overline{x}$*, which is continuous at* $\overline{p}$ *and for every* $\epsilon > 0$*, there is a neighbourhood* Q *of* $\overline{p}$ *such that*

$$|s(p') - s(p)| \leq \frac{\kappa + \epsilon}{1 - \kappa\mu} |f(p', s(p)) - f(p, s(p))| \quad \forall p', p \in Q$$

Extended Implicit Function Theorem with First-Order Approximations

Theorem (2B.9)

Let the assumptions of Robinson Theorem hold for $\mu = 0$, i.e.:

1. $f(., \overline{x})$ *is continuous at* $\overline{p}$ *and* h *a strict* **first-order approximation** *of* f *wrt.* x *at* $(\overline{p}, \overline{x})$
2. *the inverse* G^{-1} *of* $G = h + F, 0 \in G(\overline{x})$ *has a Lip-cont. svl.* σ *around 0.*

Then we **additionally** *get:*

(a) If $clm_p(f;(\overline{p},\overline{x})) < \infty$ then s is calm at $\overline{p}$ with

$$clm(s;\overline{p}) \leq lip(\sigma;0) \cdot clm_p(f;(\overline{p};\overline{x})).$$

(b) If $\widehat{lip}_p(f;(\overline{p},\overline{x})) < \infty$ then s is Lip-cont. near $\overline{p}$ with

$$lip(s;\overline{p}) \leq lip(\sigma;0) \cdot \widehat{lip}_p(f;(\overline{p};\overline{x})).$$

(c) If, along with (a), f has a first-order approximation r wrt. p at $(\overline{p},\overline{x})$, then

$$\eta(p) = \sigma(-r(p) + f(\overline{p},\overline{x})), \qquad p \in Q,$$

is a first-order approximation at $\overline{p}$ to s.

(d) If, in addition to all previous conditions, σ is affine $(\sigma(y) = \overline{x} + Ay)$ and r is strict, then η is strict and

$$\eta(p) = \overline{x} + A(-r(p) + f(\overline{p},\overline{x})), \qquad p \in Q.$$

(a) If $clm_p(f;(\overline{p},\overline{x})) < \infty$ then $clm(s;\overline{p}) \leq lip(\sigma;0) \cdot clm_p(f;(\overline{p};\overline{x}))$.

Proof.

- 2B.5 with $p := \overline{p}, \kappa = lip(\sigma;0)$ for any $\epsilon > 0$:

$$|s(p') - s(\overline{p})| \leq \frac{\kappa + \epsilon}{1 - \kappa\mu}|f(p', s(\overline{p})) - f(\overline{p}, s(\overline{p}))| \qquad \forall p' \in Q$$

- $\frac{1}{|p-\overline{p}|}$
- limsup

□

(b) analogously to (a)

(c) If $clm_p(f;(\overline{p},\overline{x})) < \infty$ and f has a first-order approx. r wrt. p, then $\eta(p) = \sigma(-r(p) + f(\overline{p},\overline{x}))$ is a first-order approx. of s

Proof.

To show: $|s(p) - \eta(p)| = o(|p - \overline{p}|)$.

1. $s(p) = \sigma(-e(p,s(p)))$ for $e(p,x) = f(p,x) - h(x)$ and $\overline{x} = s(\overline{p}) = \sigma(0) = \eta(\overline{p})$

Make Q and U (from 2B.5) smaller, such that for all $x \in U, p \in Q$ and any $\epsilon > 0$:

2. $-e(p,\overline{x})$ and $-r(p) + f(\overline{p},\overline{x})$ are close enough to zero, such that σ is Lip-cont.
3. $|e(p,x) - e(p,\overline{x})| \leq \epsilon|x - \overline{x}|$
4. $|f(p,\overline{x}) - r(p)| \leq \epsilon|p - \overline{p}|$
5. $|s(p) - s(\overline{p})| \leq \lambda|p - \overline{p}|$, $\lambda > clm(s;\overline{p})$

Utilization of Strict Differentiability

Corollary

Suppose that f is strictly differentiable at $(\overline{p}, \overline{x})$ *and that* G^{-1} *of*

$$G(x) = f(\overline{p}, \overline{x}) + \nabla_x f(\overline{p}, \overline{x})(x - \overline{x}) + F(x) = h(x) + F(x)$$

has a Lip-cont. svl. σ *around 0.*

Then*:*

- *s has a first-order approximation at* $\overline{p}$ *given by* $\eta(p) = \sigma(-\nabla_p f(\overline{p}, \overline{x})(p - \overline{p}))$
- *If* $F \equiv 0$*, then* η *is strict and*

$$\eta(p) = \overline{x} - \nabla_x f(\overline{p}, \overline{x})^{-1} \nabla_p f(\overline{p}, \overline{x})(p - \overline{p}).$$

Conclusion: *classical implicit function theorem covered by Robinson theorem.*

Extended Inverse Function Theorem with First-Order Approximations

Theorem (2B.11)

Let $f(p,x) = g(x) - p$ for some function $g : \mathbb{R}^n \to \mathbb{R}^m$ and h a strict first-order approximation to g at $\bar{x}$.

Then: *$(g+F)^{-1}$ has a Lip-cont. svl. s around $\bar{p}$, if and only if $(h+F)^{-1}$ has such a localization σ.*

Moreover, σ is a first-order approximation of s at $\bar{p}$, and

$$lip(s;\bar{p}) = lip(\sigma;\bar{p}).$$

Ample parametrizations

Consider the equation

$$0 \in g(x) + F(x). \tag{1}$$

Then a function $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ is called

- a **parametrization**, if $f(\bar{p}, x) \equiv g(x)$ for a particular $\bar{p}$ and it is
- **ample at** $\bar{x}$, if $\nabla_p f(\bar{p}, \bar{x})$ has full rank $(rk = m)$.
- associated solution mapping

$$S : p \mapsto \{x | f(p, x) + F(x) \ni 0\} \tag{2}$$

- supplementary parameters to ensure rank condition

Equivalences from Ampleness

Theorem (2C.2)

Let f be an ample parametrization at $(\bar{p}, \bar{x})$ of g as before and let h a be strict first-order approximation of f wrt. x unif. in p at $(\bar{p}, \bar{x})$

Then *the following statements are equivalent:*

(a) *S as in (2) has a Lip-cont. svl. around $\bar{p}$ for $\bar{x}$;*

(b) *$(h + F)^{-1}$ has a Lip-cont. svl. around 0 for $\bar{x}$;*

(c) *$(g + F)^{-1}$ has a Lip-cont. svl. around 0 for $\bar{x}$.*

(d) *$\tilde{S}$ has a Lip-cont. svl. around $(\bar{p}, 0)$ for $\bar{x}$.*

Equivalences from Ampleness

(a) S as in (2) has a Lip-cont. svl. around $\overline{p}$ for $\overline{x}$;

(b) $(h+F)^{-1}$ has a Lip-cont. svl. around 0 for $\overline{x}$;

(c) $(g+F)^{-1}$ has a Lip-cont. svl. around 0 for $\overline{x}$.

(d) $\tilde{S}$ has a Lip-cont. svl. around $(\overline{p}, 0)$ for $\overline{x}$.

Proof.

- $(b) \implies (a),(d)$ with Robinson Theorem (2B.5)
- $(b) \Leftrightarrow (c)$ from the Extended Inverse Function Theorem (2B.11)
- $(d) \implies (a)$ by setting the supplementary parameters to zero
- $(a) \implies (b)$ by next Lemma (2C.1) and Contraction Mapping Principle applied to ψ

□

Local Selection from Ampleness

Lemma (2C.1)

Let $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m$ *be an ample parametrization of (1) at* $\overline{x}$*. If f has a strict first-order approximation h wrt. x at* $(\overline{p}, \overline{x})$*.*

Then:

$$\Psi : (x, y) \mapsto \{p | e(p, x) + y = 0\}$$

with $e(p, x) = f(p, x) - h(x)$ *has a local selection* ψ *around* $(\overline{x}, 0)$ *for* $\overline{p}$*.*

Furthermore:

$$\widehat{lip}_x(\psi; (\overline{x}, 0)) = 0$$

and

$$\widehat{lip}_y(\psi; (\overline{x}, 0)) < \infty.$$

Parametric Robustness

Theorem (2C.3)

Let g in (1) be strictly differentiable and h the linearisation of g:

$$h(x) = g(\bar{x}) + \nabla g(\bar{x})(x - \bar{x}).$$

Then *the following statements are equivalent:*

1. $(h + F)^{-1}$ *has a Lip-cont. svl. around 0 for* $\bar{x}$*;*
2. *For every parametrization f of (1) that is strictly differentiable at $(\bar{x}, \bar{p})$, S as in (2) has a Lip-cont. svl. around $\bar{p}$ for $\bar{x}$.*

Semiderivatives

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **(strictly) semidifferentiable** at $\overline{x}$, if it has a (strict) first-order approximation h at $\overline{x}$ of the form

$$h(x) = f(\overline{x}) + \phi(x - \overline{x}),$$

where ϕ is continuous and positive homogeneous.

Definition

A function f is (one-sided) directionally differentiable, if

$$f'(\overline{x}; w) := \lim_{t \searrow 0} \frac{f(\overline{x} + tw) - f(\overline{x})}{t}$$

exists for all $w \in \mathbb{R}^n$.

Semiderivatives

Definition

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$$h(x) = f(\overline{x}) + \phi(x - \overline{x}),$$

where ϕ is continuous and positive homogeneous.

Definition

A function f is semidifferentiable, if and only if

$$Df(\overline{x})(w) := \lim_{\substack{t \searrow 0 \\ w' \to w}} \frac{f(\overline{x} + tw') - f(\overline{x})}{t}$$

exists for all $w \in \mathbb{R}^n$.

Semiderivatives

Lemma

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is semidifferentiable at $\overline{x}$, then it is directionally differentiable at $\overline{x}$ and

$$Df(\overline{x})(w) = f'(\overline{x}; w)$$

If $lip(f; \overline{x}) < \infty$, then directional differentiability implies semidifferentiability.

Semiderivatives

Properties

- *the semiderivative* $Df(\overline{x}) := \phi$ *is unique*
- *semidifferentiable* $\implies$
  - *directionally diff. (equivalent if* $lip(f;\overline{x}) < \infty$*)*
  - $Df(\overline{x})(w) = f'(\overline{x}; w)$
  - $clm(f;\overline{x}) < \infty$ *(strictly* $\implies$ $lip < \infty$*)*
- *chain rule: if*
  - f *is (strictly) semidiff. at* $\overline{x}$
  - g *is Lip-cont. and semidiff. (strictly diff.) at* $\overline{y} := f(\overline{x})$

then $g \circ f$ *is (strictly) semidiff. at* $\overline{x}$ *and* $D(g \circ f)(\overline{x}) = Dg(\overline{y}) \circ Df(\overline{x})$

Semiderivatives

Examples

- $f(x) = e^{|x|}$: not diff. at 0, but strictly semidiff. with $Df(0) : w \mapsto |w|$
- $f(x, y) = min\{x, y\}$: not diff. along $x = y$, but strictly semidiff. with $Df(x, x)(w_1, w_2) = min\{w_1, w_2\}$

Implicit Function Theorem Utilizing Semiderivatives

Theorem (2D.6)

Let $\overline{x} \in S(\overline{p})$ *for* $S : p \mapsto \{x | f(p, x) + F(x) \ni 0\}$, *f be strictly semidifferentiable at* $(\overline{p}, \overline{x})$ *and assume that* G^{-1} *of the mapping*

$$G(x) = f(\overline{p}, \overline{x}) + D_x f(\overline{p}, \overline{x})(x - \overline{x}) + F(x), \quad 0 \in G(\overline{x})$$

has a Lip-cont. svl. σ *around* 0 *for* $\overline{x}$, *which is semidifferentiable at 0.*

Then:

S has a Lip-cont. svl. s around $\overline{p}$ *for* $\overline{x}$, *which is semidiff. at* $\overline{p}$ *and*

$$Ds(\overline{p}) = D\sigma(0) \circ (-D_p f(\overline{p}, \overline{x}))$$

Implicit Function Theorem Utilizing Semiderivatives

Proof.

- strictly semidiff. $\implies lip(f) < \infty \implies \exists$Lip-cont. svl. s (2B.5, 2B.9b) and
- $r(p) := f(\overline{x}, \overline{p}) + D_p f(\overline{p}, \overline{x})(p - \overline{p})$ is a strict 1st-order approx.
- $s(\overline{p}) = \sigma(0) = \overline{x}$
- to show: $s(\overline{p}) + Ds(\overline{p})(p - \overline{p})$ is a 1st-order approx. to s. (clm(e) $= 0$)
- $|s(p) - s(\overline{p}) - (D\sigma(0) \circ (-D_p f(\overline{p}, \overline{x})))(p - \overline{p})| = o(|p - \overline{p}|) \implies clm = 0$

□

Piecewise Smooth Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **piecewise smooth** on an open set O, if it is continuous on O and for each $x \in O$ there is a finite collection $\{f_i\}_{i \in \mathcal{I}}$ of C^1-functions defined on a neighbourhood of x such that

$$f(y) \in \{f_i(y) | i \in \mathcal{I}\}, \text{when } |y - x| < \epsilon$$

for some ϵ.

Definition

$\mathcal{I}(x) := \{i \in \mathcal{I} | f(x) = f_i(x)\}$ is called a **local representation** of f at x.

It is called **minimal**, if no subset of $\mathcal{I}(x)$ is a local representation.

Piecewise Smooth Functions

Lemma (Decomposition)

Let f be piecewise smooth with minimal representation at $\overline{x}$.

Then for each $i \in \mathcal{I}(\overline{x})$ there exists an open set O_i, such that $\overline{x} \in \overline{O_i}$ and $f(x) = f_i(x)$ on O_i.

Lemma (Semidifferentiablity)

If f is piecewise smooth, then f is semidifferentiable.

Moreover, $Df(\overline{x})$ is piecewise smooth. If $\mathcal{I}(\overline{x})$ is minimal, then the local representation of $Df(\overline{x})$ is given by $\{Df_i(\overline{x})\}_{i\in\mathcal{I}(\overline{x})}$

Applications in Optimization

Example (Piecewise Smoothness of Special Projection Mappings)

Let $C := \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, \ldots, m\}$ with g_i convex C^2-functions and $\overline{x} \in C$ such, that for the active constraints (i.e. $g_i(\overline{x}) = 0$) it holds that $\nabla g_i(\overline{x})$ are linearly independent.

Then the projection mapping P_C is piecewise smooth in a neighbourhood of $\overline{x}$.

Example (Projection Mapping)

Let $C := \{x \in \mathbb{R}^n | Ax = b \in \mathbb{R}^m\}$. If the rows of A are linearly independent, then

$$P_C(x) = (I - A^T(AA^T)^{-1}A)x + A^T(AA^T)^{-1}b$$