Variational Inequalities with Polyhedral Convexity

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Outline

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1. Recap
2. Polyhedral Convex Sets
3. Localization under Polyhedral Convexity

Previously in our seminar...

We study *parametrized generalized equations* of the form

$$f(p,x) + F(x) \ni 0$$

where $f : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.

Consider properties of solution mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$,

$$S : p \mapsto \{x | f(p,x) + F(x) \ni 0\}.$$

Special case: Let $C \subset \mathbb{R}^n$ convex, closed. The *variational inequality*

$$x \in C, \quad \langle f(p,x), x' - x \rangle \geq 0 \; \forall x' \in C$$

is equivalent to the generalized equation

$$f(p,x) + N_C(x) \ni 0$$

Previously in our seminar...

Theorem 2B.1

Theorem (Robinson Implicit Function Theorem)

For the solution mapping S to a parameterized variational inequality, consider a pair $(\bar{p}, \bar{x})$ with $\bar{x} \in S(\bar{p})$. Assume that:

- *$f(p, x)$ is differentiable with respect to x in a neighbourhood of the point $(\bar{p}, \bar{x})$, and both $f(p, x)$ and $\nabla_x f(p, x)$ depend continuously on (p, x) in this neighbourhood;*
- *the inverse G^{-1} of the set valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by*

$$G(x) = f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x})(x - \bar{x}) + N_C(x), \text{ with } G(\bar{x}) \ni 0,$$

has a Lipschitz continuous single-valued localization σ around 0 for $\bar{x}$ with

$$lip(\sigma; 0) \leq \kappa.$$

Then S has a single-valued localization s around $\bar{p}$ for $\bar{x}$ which is continuous at $\bar{p}$, and moreover for every $\epsilon > 0$ there is a neighbourhood Q of $\bar{p}$ such that

$$\left|s(p') - s(p)\right| \leq (\kappa + \epsilon) \left|f(p', s(p)) - f(p, s(p))\right| \text{ for all } p', p \in Q.$$

Wolfgang Stockinger November 24, 2015 17 / 24

Previously in our seminar...

Several Extensions:

- Theorem 2B.5:
  F general set-valued mapping
  f not necessarily differentiable
- Corollary 2B.10:
  F general set-valued mapping
  f strictly differentiable

Today:

- $F = N_C$ (variational inequalities)
- f strictly differentiable

Solution Mappings for Parametrized Variational Inequalities

Theorem (2E.1)

For a variational inequality and its solution mapping, let $\bar{p}$ and $\bar{x}$ be such that $\bar{x} \in S(\bar{p})$. Assume that

(a) *f is strictly differentiable at $(\bar{p}, \bar{x})$;*

(b) *the inverse G^{-1} of the mapping*

$$G(x) = f(\bar{p}, \bar{x}) + \nabla_x f(\bar{p}, \bar{x})(x - \bar{x}) + N_C(x), \quad \text{with } G(\bar{x}) \ni 0,$$

has a Lipschitz continuous single-valued localization σ around 0 for $\bar{x}$.

Then S has a Lipschitz continuous single-valued localization s around $\bar{p}$ for $\bar{x}$ with

$$lip(s; \bar{p}) \leq lip(\sigma; 0) \cdot |\nabla_p f(\bar{p}, \bar{x})|,$$

and this localization s has a first-order approximation η at $\bar{p}$ given by

$$\eta(p) = \sigma(-\nabla_p f(\bar{p}, \bar{x})(p - \bar{p})).$$

Solution Mappings for Parametrized Variational Inequalities

Theorem (2E.1 (cont'd))

Moreover, under the ample parametrization condition

$$\operatorname{rank} \nabla_p f(\bar{p}, \bar{x}) = n,$$

the existence of a Lipschitz continuous single-valued localization s of S around $\bar{p}$ for $\bar{p}$ not only follows from, but also necessitates the existence of a localization σ of G^{-1} having the properties described.

Proof.

Application of Corollary 2B.10 and Theorem 2C.2. □

Goal for today:

Understand better the circumstances in which existence of s.v.l. σ of G^{-1} around 0 for $\bar{x}$ as assumed in (b) of Theorem 2E.1 is assured for special case where C polyhedral, convex

Recap: Cones

Let $C \subset \mathbb{R}^n$ convex, $x \in C$.

Definition (Normal Cone)

$$N_C(x) = \{v | \langle v, x' - x \rangle \leq 0 \ \forall x' \in C\} \text{ (closed, convex)}$$

Definition (Polar Cone)

Let K be a closed convex cone in $\mathbb{R}^n$. Then

$$K^* = \{y | \langle y, x \rangle \leq 0 \ \forall x \in K\}$$

is the polar cone to K. The polar cone K^* is closed and convex. In particular

$$y \in N_K(x) \iff x \in N_{K^*}(y) \iff x \in K, y \in K^*, \langle x, y \rangle = 0$$

Definition (Tangent Cone)

$$T_C(x) = \{v | v = \lim \frac{1}{\tau^k}(x^k - x) \text{ for some } x^k \to x, x^k \in C, \tau^k \searrow 0\}$$

We have the relations

$$T_C(x) = N_C(x)^* \quad \text{and} \quad N_C(x) = T_C(x)^*$$

Recap: Differentiability

Definition (Strict Differentiability)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **strictly differentiable** at point $\overline{x}$ if there is a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\text{lip}(e; \overline{x}) = 0 \text{ for } e(x) = f(x) - [f(\overline{x}) + A(x - \overline{x})].$$

Definition ((strict) semidifferentiablility)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **(strictly) semi-differentiable** at point $\overline{x}$ if it has a (strict) first-order approximation h at $\overline{x}$ of the form

$$h(x) = f(\overline{x}) + \phi(x - \overline{x}),$$

where ϕ is continuous and positive homogeneous.

Outline

Polyhedral Convex Sets

Investigate existence of Lipschitz continuous s.v.l. σ around 0 for $\overline{x}$ of G^{-1} where

$$G(x) = f(\overline{p}, \overline{x}) + \nabla_x f(\overline{p}, \overline{x})(x - \overline{x}) + N_C(x), \quad \text{with } G(\overline{x}) \ni \overline{x}$$

where $\overline{x} \in S(\overline{p})$. We analyze the local geometry of $gph\, N_C$ for the special case of **polyhedral convex sets** C.

Definition

A set C in $\mathbb{R}^n$ is said to be **polyhedral convex** when it can be expressed as the intersection of finitely many closed half-spaces

Properties of polyhedral convex sets C:

- $C = \{x | \langle b_i, x \rangle \leq \alpha_i \text{ for } i = 1, \dots, m\}$
- C is closed
- $\alpha_i = 0\ \forall i = 1, \dots, m \Rightarrow$ C is **cone**

Polyhedral Convex Sets

Theorem (2E.2: Minkowski-Weyl Theorem)

A set $K \subset \mathbb{R}^n$ is a polyhedral convex cone **if and only if** *there is a collection of vectors $b_1, \ldots, b_m$ such that*

$$K = \{y_1 b_1 + \cdots + y_m b_m | y_i \geq 0 \text{ for } i = 1, \ldots, m\}.$$

Polyhedral Convex Sets

Theorem (2E.3: Variational Geometry of Polyhedral Convex Sets)

Let $C = \{x | \langle b_i, x\rangle \le \alpha_i \text{ for } i = 1, \dots, m\}$ *be a polyhedral convex set. Let* $x \in C$ *and* $I(x) = \{i | \langle b_i, x\rangle = \alpha_i\}$, *this being the set of constraints that are active at* x. *Then the tangent and normal cone to* C *at* x *are polyhedral convex, with the tangent cone having the representation*

$$T_C(x) = \{w | \langle b_i, w\rangle \le 0 \text{ for } i \in I(x)\}$$

and the normal cone having the representation

$$N_C(x) = \left\{ v \,\middle|\, v = \sum_{i=1}^{m} y_i b_i \text{ with } y_i \ge 0 \text{ for } i \in I(x), y_i = 0 \text{ for } i \notin I(x) \right\}.$$

Furthermore, the tangent cone has the properties that

$$W \cap [C - x] = W \cap T_C(x) \text{ for some neighborhood } W \text{ of } 0$$

and

$$T_C(x) \supset T(\bar{x}) \text{ for all } x \text{ in some neighborhood } U \text{ of } \bar{x}.$$

Critical Cone

Definition (Critical Cone)

For a convex set C, any $x \in C$ and any $v \in N_C(x)$, the critical cone to C at x for v is

$$K_C(x, v) = \{w \in T_C(x) | w \perp v\} .$$

Note: C polyhedral implies $K_C(x, v)$ polyhedral

Reduction Lemma

Lemma (Reduction Lemma)

Let C be a polyhedral convex set in $\mathbb{R}^n$, and let

$$\bar{x} \in C, \quad \bar{v} \in N_C(\bar{x}), \quad K = K_C(\bar{x}, \bar{v}).$$

The graphical geometry of the normal cone mapping N_C around $(\bar{x}, \bar{v})$ reduces then to the graphical geometry of the normal cone mapping N_K around $(0, 0)$, in the sense that

$$O \cap [\text{gph } N_C - (\bar{x}, \bar{v})] = O \cap \text{gph } N_K \ \text{ for some neighborhood } O \text{ of } (0, 0).$$

In other words, one has

$$\bar{v} + u \in N_C(\bar{x} + w) \Leftrightarrow u \in N_K(w) \text{ for } (w, u) \text{ sufficiently near to } (0, 0).$$

Proof:
Blackboard.

Critical Subspaces

Definition (Critical Subspaces)

The smallest linear subspace that includes the critical cone $K_C(x,v)$ will be denoted by $K_C^+(x,v)$, whereas the largest linear subspace that is included in $K_C(x,v)$ will be denoted by $K_C^-(x,v)$, the formulas being

$$K_C^+(x,v) = K_C(x,v) - K_C(x,v) = \{w - w' | w, w' \in K_C(x,v)\},$$
$$K_C^-(x,v) = K_C(x,v) \cap [-K_C(x,v)] = \{w \in K_C(x,v) | -w \in K_C(x,v)\}.$$

Outline

Affine-Polyhedral Variational Inequalities

Theorem (Affine-Polyhedral Variational Inequalities)

For an affine function $x \mapsto a + Ax$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, and a polyhedral convex set $C \subset \mathbb{R}^n$, consider the variational inequaility

$$a + Ax + N_C(x) \ni 0.$$

Let $\bar{x}$ be a solution and let $\bar{v} = -a - A\bar{x}$, so that $\bar{v} \in N_C(\bar{x})$, and let $K = K_C(\bar{x}, \bar{v})$ be the associated critical cone. Then for the mappings

$$G(x) = a + Ax + N_C(x) \text{ with } G(\bar{x}) \ni 0,$$
$$G_0(w) = Aw + N_K(w) \text{ with } G_0(0) \ni 0,$$

the following properties are equivalent:

(a) *G^{-1} has a Lipschitz continuous single-valued localization σ around 0 for $\bar{x}$;*

(b) *G_0^{-1}* **is a single-valued** *mapping with all of $\mathbb{R}^n$ as its domain,*

in which case G_0^{-1} is necessarily Lipschitz continuous globally and the function $\sigma(v) = \bar{x} + G_0^{-1}(v)$ furnishes the localization in (a). Moreover, in terms of critical subspaces $K^+ = K_C^+(\bar{x}, \bar{v})$ and $K^- = K_C^-(\bar{x}, \bar{v})$, the following condition is **sufficient** *for (a) and (b) to hold:*

$$w \in K^+, \quad Aw \perp K^-, \quad \langle w, Aw \rangle \leq 0 \Longrightarrow w = 0. \qquad (20)$$

Example

Examples

1. When the critical cone K is a subspace, the condition in (20) reduces to the nonsingularity of the linear transformation $K \ni w \mapsto P_K(Aw)$, where P_K is the projection onto K.

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2. When the critical cone K is pointed, in the sense that $K \cap (-K) = \{0\}$, the condition in (20) reduces to the requirement that $\langle w, Aw \rangle > 0$ for all nonzero $w \in K^+$

Example

Examples

1. When the critical cone K is a subspace, the condition in (20) reduces to the nonsingularity of the linear transformation $K \ni w \mapsto P_K(Aw)$, where P_K is the projection onto K.
2. When the critical cone K is pointed, in the sense that $K \cap (-K) = \{0\}$, the condition in (20) reduces to the requirement that $\langle w, Aw \rangle > 0$ for all nonzero $w \in K^+$
3. Condition (20) always holds when A is the identity matrix.

Localization Criterion under Polyhedral Convexity

Theorem (Localization Criterion under Polyhedral Convexity)

For a variational inequality and its solution mapping under the assumption that C is polyhedral convex and f is strictly differentiable at $(\bar{p}, \bar{x})$, with $\bar{x} \in S(\bar{p})$, let

$$A = \nabla_x f(\bar{p}, \bar{x}) \quad \text{and} \quad K = K_C(\bar{x}, \bar{v}) \text{ for } \bar{v} = -f(\bar{p}, \bar{x}).$$

Suppose that for each $u \in \mathbb{R}^n$ there is a unique solution $w = \bar{s}(u)$ to the auxiliary variational inequality $Aw - u + N_K(w) \ni 0$, this being equivalent to saying that

$$\bar{s} = (A + N_K)^{-1} \text{ is everywhere single-valued,} \qquad (29)$$

in which case the mapping $\bar{s}$ is Lipschitz continuous globally. (A sufficient condition for this assumption to hold is the property in (20) with respect to the critical subspaces $K^+ = K_C^+(\bar{x}, \bar{v})$ and $K^- = K_C^-(\bar{x}, \bar{v})$.)
Then S has a Lipschitz continuous single-valued localization s around $\bar{p}$ for $\bar{x}$ which is semidifferentiable with

$$\operatorname{lip}(s; \bar{p}) \le \operatorname{lip}(\bar{s}; 0)|\nabla_p f(\bar{p}, \bar{x})|, \quad Ds(\bar{p})(q) = \bar{s}(-\nabla_p f(\bar{p}, \bar{x})q).$$

Moreover, under the ample parametrization condition, $\operatorname{rank} \nabla_p f(\bar{p}, \bar{x}) = n$, condition (29) is not only sufficient, but also necessary for a Lipschitz continuous single-valued localization of S around $\bar{p}$ for $\bar{x}$.

Local Behavior of Critical Cones and Subspaces

Theorem (Local Behavior of Critical Cones and Subspaces)

Let $C \subset \mathbb{R}^n$ be a polyhedral convex set, and let $\bar{v} \in N_C(\bar{x})$. Then the following properties hold:

(a) $K_C(x, v) \subset K_C^+(\bar{x}, \bar{v})$ *for all* $(x, v) \in gphN_C$ *in some neighborhood of* $(\bar{x}, \bar{v})$;

(b) $K_C(x, v) = K_C^+(\bar{x}, \bar{v})$ *for some* $(x, v) \in gphN_C$ *in each neighborhood of* $(\bar{x}, \bar{v})$.

Thank you for your attention!