Set-Valued Analysis of Solution Mappings

from Chapter 3 of "Implicit Functions and Solution Mappings" by A.L.Dontchev and R.T.Rockafellar

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Introduction

Chapter 2:

Solution mappings for parameter dependent problems:
$S(p)$ set of all x satisfying $f(p,x)=0 \quad (f:\mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^m)$
with single-valued localization.
Parametrized general equation:

$$f(p,x)+F(x) \ni 0 \qquad \text{with F a constant mapping}$$
$$F(x) \equiv -K \qquad (K = \mathbb{R}^s_- \times \{0\}^{m-s}).$$

Chapter 3:

$F(x)$ is not a normal cone mapping N_C.
Therefore single-valued localizations are unlikely to exist. We are confronted with a "varying set" $S(p)$ which cannot be reduced to a "varying point".
We are rather looking for a *implicit mapping theorem instead of a generalized implicit function theorem.*

1. 3.1 Set Convergence
2. 3.2 Continuity of Set-Valued Mappings
3. 3.3 Lipschitz Continuity of Set-Valued Mappings
4. 3.4 Outer Lipschitz Continuity

Inner and Outer Limits

Consider a sequence $\{C^k\}_{k=1}^{\infty}$ of subsets in $\mathbb{R}^n$

(a) outer limit:
$\limsup_k C^k$ is the set of all $x \in \mathbb{R}^n$ for which there exists

$$N \in \mathcal{N}^{\#} \text{ and } x^k \in C^k \text{ for } k \in N \text{ such that } x^k \overset{N}{\to} x$$

(b) inner limit:
$\liminf_k C^k$ is the set of all $x \in \mathbb{R}^n$ for which there exists

$$N \in \mathcal{N} \text{ and } x^k \in C^k \text{ for } k \in N \text{ such that } x^k \overset{N}{\to} x$$

(c) limit:

$$C = \lim_k C^k = \limsup_k C^k = \liminf_k C^k$$

In this case C^k is said to converge to C in the sense of Painlevé-Kuratowski convergence.

Inner and Outer Limits by neighborhoods

Consider a sequence $\{C^k\}_{k=1}^{\infty}$ of subsets in $\mathbb{R}^n$

(a) $\limsup_{k\to\infty} C^k := \{x \mid \forall \varepsilon > 0, \exists N \in \mathcal{N}^{\#} : x \in C^k + \varepsilon\mathbb{B}\,(k \in N)\}$

(b) $\liminf_{k\to\infty} C^k := \{x \mid \forall \varepsilon > 0, \exists N \in \mathcal{N} : x \in C^k + \varepsilon\mathbb{B}\,(k \in N)\}$

Both outer and inner limits are *closed sets.*

Distance function

From Section 1.4: Distance of a point $x \in \mathbb{R}^n$ to a subset C

$$d_C(x) = d(x, C) = \inf_{y \in C} |x - y|.$$

Proposition 3A.1 (Distance Function Characterizations of Limits)

(a) $\limsup_{k\to\infty} C^k = \{x \mid \liminf_{k\to\infty} d(x, C^k) = 0\}$

(b) $\liminf_{k\to\infty} C^k = \{x \mid \lim_{k\to\infty} d(x, C^k) = 0\}$

Painlevé-Kuratowski Convergence

Theorem 3A.2 (Characterization of Painlevé-Kuratowski Convergence)

For a sequence C^k of sets in $\mathbb{R}^n$ and a closed set $C \subset \mathbb{R}^n$ one has:

(a) $C \subset \liminf_k C^k$ if and only if for every open set $O \subset \mathbb{R}^n$ with $C \cap O \neq \emptyset$ there exists $N \in \mathcal{N}$ such that $C^k \cap O \neq \emptyset \; \forall k \in N$;

(b) $C \supset \limsup_k C^k$ if and only if for every compact set $B \subset \mathbb{R}^n$ with $C \cap B = \emptyset$ there exists $N \in \mathcal{N}$ such that $C^k \cap B = \emptyset$ $\forall k \in N$;

Proof.

Painlevé-Kuratowski Convergence

Theorem 3A.2 (Characterization of Painlevé-Kuratowski Convergence)

For a sequence C^k of sets in $\mathbb{R}^n$ and a closed set $C \subset \mathbb{R}^n$ one has:

(c) $C \subset \liminf_k C^k$ if and only if for every $\rho > 0$ and $\varepsilon > 0$ there is an index set $N \in \mathcal{N}$ such that $C \cap \rho\mathbb{B} \subset C^k + \varepsilon\mathbb{B}\ \forall k \in N$;

(d) $C \supset \limsup_k C^k$ if and only if for every $\rho > 0$ and $\varepsilon > 0$ there is an index set $N \in \mathcal{N}$ such that $C^k \cap \rho\mathbb{B} \subset C + \varepsilon\mathbb{B}\ \forall k \in N$;

Painlevé-Kuratowski Convergence

Theorem 3A.2 (Characterization of Painlevé-Kuratowski Convergence)

For a sequence C^k of sets in $\mathbb{R}^n$ and a closed set $C \subset \mathbb{R}^n$ one has:

(e) $C \subset \liminf_k C^k$ if and only if $\limsup_k d(x, C^k) \le d(x, C)$ $\forall x \in \mathbb{R}^n$;

(f) $C \supset \limsup_k C^k$ if and only if $d(x, C) \le \liminf_k d(x, C^k)$ $\forall x \in \mathbb{R}^n$;

Proof.

Excess and Pompeiu-Hausdorff Distance

For sets C and D in $\mathbb{R}^n$, the excess of C beyond D is defined by

$$e(C, D) = \sup_{x \in C} d(x, D),$$

where the convention is used that

$$e(\emptyset, D) = \begin{cases} 0 & \text{when } D \neq \emptyset, \\ \infty & \textit{otherwise}. \end{cases}$$

The Pompeiu-Hausdorff distance between C and D is the quantity

$$h(C, D) = \max\{e(C, D), e(D, C)\}.$$

Excess and Pompeiu-Hausdorff Distance

Proposition 3A.3 (Characterization of Pompeiu-Hausdorff Distance)

For any nonempty sets C an D in $\mathbb{R}^n$, one has

$$h(C, D) = \sup_{x \in \mathbb{R}^n} |d(x, C) - d(x, D)|.$$

Proof.

Excess and Pompeiu-Hausdorff Distance

Pompeiu-Hausdorff Convergence

A sequence of sets $\{C^k\}_{k=1}^{\infty}$ is said to converge with respect to the Pompeiu-Hausdorff distance to a set C when C is closed and $h(C^k, C) \to 0$ as $k \to \infty$.

Pompeiu-Hausdorff versus Painlevé-Kuratowski

Theorem 3A.4 (Pompeiu-Hausdorff versus Painlevé-Kuratowski)

If a sequence of closed sets $\{C^k\}_{k=1}^{\infty}$ converges to C with respect to Pompeiu-Hausdorff distance then it also converges to C in Painlevé-Kuratowski sense.
The opposite implication holds if there is a bounded set X which contains C and every C^k.

Proof.

Conditions for Pompeiu-Hausdorff Convergence

Theorem 3A.6 (Conditions for Pompeiu-Hausdorff Convergence)

A sequence C^k of sets in $\mathbb{R}^n$ is convergent with respect to Pompeiu-Hausdorff distance to a closed set $C \subset \mathbb{R}^n$ if both of the following conditions hold:

(a) for every open set $O \subset \mathbb{R}^n$ with $C \cap O \neq \emptyset$ there exists $N \in \mathcal{N}$ such that $C^k \cap O \neq \emptyset$ for all $k \in N$;

(b) for every open set $O \subset \mathbb{R}^n$ with $C \subset O$ there exists $N \in \mathcal{N}$ such that $C^k \subset O$ for all $k \in N$;

Moreover, condition (a) is always necessary for Pompeiu-Hausdorff convergence, while (b) is necessary when the set C is bounded.

$\rightarrow$ Unbounded Issue

3.2 Continuity of Set-Valued Mappings

Inner and outer limit for set-valued mappings:

$$\limsup_{y\to\bar{y}} S(y) = \bigcup_{y^k\to\bar{y}} \limsup_{k\to\infty} S(y^k)$$
$$= \left\{x \middle|\ \exists y^k \to \bar{y}, \exists x^k \to x \text{ with } x^k \in S(y^k)\right\}$$

and

$$\liminf_{y\to\bar{y}} S(y) = \bigcap_{y^k\to\bar{y}} \liminf_{k\to\infty} S(y^k)$$
$$= \left\{x \middle|\ \forall y^k \to \bar{y}, \exists N \in \mathcal{N}, x^k \stackrel{N}{\to} x \text{ with } x^k \in S(y^k)\right\}$$

Semicontinuity and Continuity

Semicontinuity and Continuity

A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semicontinuous (osc) at $\bar{y}$ when

$$\limsup_{y \to \bar{y}} S(y) \subset S(\bar{y})$$

and inner semicontinuous (isc) at $\bar{y}$ when

$$\liminf_{y \to \bar{y}} S(y) \supset S(\bar{y})$$

It is called Painlevé-Kuratowski continuous at $\bar{y}$ when it is both osc and isc at $\bar{y}$, as expressed by

$$\lim_{y \to \bar{y}} S(y) = S(\bar{y})$$

S is called Pompeiu-Hausdorff continuous at $\bar{y}$ when

$$S(\bar{y}) \text{ is closed and } \lim_{y \to \bar{y}} h(S(y), S(\bar{y})) = 0.$$

Theorem 3B.2 (Characterization of Semicontinuity)

For $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, a set $D \subset \mathbb{R}^m$ and $\bar{y} \in domS$ we have:

(a) S is osc at $\bar{y}$ relative to D if and only if for every $x \notin S(\bar{y})$ there are neighborhoods U of x and V of $\bar{y}$ such that $D \cap V \cap S^{-1}(U) = \emptyset$;

(b) S is isc at $\bar{y}$ relative to D if and only if for every $x \in S(\bar{y})$ and every neighborhood U of x there exists a neighborhood V of $\bar{y}$ such that $D \cap V \subset S^{-1}(U)$;

Proof.

Characterization of Semicontinuity

Theorem 3B.2 (Characterization of Semicontinuity)

For $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, a set $D \subset \mathbb{R}^m$ and $\bar{y} \in domS$ we have:

(c) S is osc at every $y \in domS$ if and only if gph S is closed;

(d) S is osc relative to a set $D \subset \mathbb{R}^m$ if and only if $S^{-1}(B)$ is closed relative to D for every compact set $B \subset \mathbb{R}^n$;

(e) S is isc relative to a set $D \subset \mathbb{R}^m$ if and only if $S^{-1}(O)$ is open relative to D for every open set $O \subset \mathbb{R}^n$;

Proof.

Theorem 3B.2 (Characterization of Semicontinuity)

For $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, a set $D \subset \mathbb{R}^m$ and $\bar{y} \in domS$ we have:

(f) S is osc at $\bar{y}$ relative to a set $D \subset \mathbb{R}^m$ if and only if the distance function $y \mapsto d(x, S(y))$ is lower semicontinuous at $\bar{y}$ relative to D for every $x \in \mathbb{R}^n$;

(g) S is isc at $\bar{y}$ relative to a set $D \subset \mathbb{R}^m$ if and only if the distance function $y \mapsto d(x, S(y))$ is upper semicontinuous at $\bar{y}$ relative to D for every $x \in \mathbb{R}^n$;

Thus, S is continuous relative to D at $\bar{y}$ if and only if the distance function $y \mapsto d(x, S(y))$ is continuous at $\bar{y}$ relative to D for every $x \in \mathbb{R}^n$;

Characterization of Pompeiu-Hausdorff Continuity

Theorem 3B.3 (Characterization of Pompeiu-Hausdorff Continuity)

A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is Pompeiu-Hausdorff continuous at $\bar{y}$ if $S(\bar{y})$ is closed and both of the following conditions hold:

(a) for every open set $O \subset \mathbb{R}^n$ with $S(\bar{y}) \cap O \neq \emptyset$ there exists a neighborhood V of $\bar{y}$ such that $S(y) \cap O \neq \emptyset$ for all $y \in V$;

(b) for every open set $O \subset \mathbb{R}^n$ with $S(\bar{y}) \subset O$ there exists a neighborhood V of $\bar{y}$ such that $S(y) \subset O$ for all $y \in V$.

Characterization of Pompeiu-Hausdorff Continuity

Theorem 3B.3 (Characterization of Pompeiu-Hausdorff Continuity)

Moreover, if S is Pompeiu-Hausdorff continuous at $\bar{y}$, then it is continuous at $\bar{y}$.
On the other hand, when $S(\bar{y})$ is nonempty and bounded, Pompeiu-Hausdorff continuity of S at $\bar{y}$ reduces to continuity together with the existence of a neighborhood V of $\bar{y}$ such that $S(V)$ is bounded;
in this case conditions (a) and (b) are not only sufficient but also necessary for continuity of S at $\bar{y}$.

Applications in Optimization

Consider the following minimization problem with parameter $p \in P \subset \mathbb{R}^d$, the objective function $f_0 : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$, and the feasible set mapping $S_{feas} : P \rightrightarrows \mathbb{R}^n$:

$$\text{minimize } f_0(p,x) \text{ over all } x \in \mathbb{R}^n \text{ satisfying } x \in S_{feas}(p).$$

then the optimal value mapping acting from $\mathbb{R}^d$ to $\mathbb{R}$ is defined by:

$$S_{val} : p \mapsto \inf_x \{f_0(p,x) | x \in S_{feas}(p)\}$$

and the optimal set mapping acting from P to $\mathbb{R}^n$

$$S_{opt} : p \mapsto \{x \in S_{feas}(p) | f_0(p,x) = S_{val}(p)\}.$$

Basic Continuity Properties of Solution Mappings in Optimization

Theorem 3B.5 (Basic Continuity Properties of Solution Mappings in Optimization)

In the preceding notation, let $\bar{p} \in P$ be fixed with the feasible set $S_{feas}(\bar{p})$ nonempty and bounded, and suppose that:

(a) the mapping S_{feas} is Pompeiu-Hausdorff continuous at $\bar{p}$ relative to P, or equivalently, S_{feas} is continuous at $\bar{p}$ relative to P with $S_{feas}(Q \cap P)$ bounded for some neighborhood Q of $\bar{p}$,

(b) the function f_0 is continuous relative to $P \times \mathbb{R}^n$ at $(\bar{p}, \bar{x})$ for every $\bar{x} \in S_{feas}(\bar{p})$.

Then the optimal value mapping S_{val} is continuous at $\bar{p}$ relative to P, whereas the optimal set mapping S_{opt} is osc at $\bar{p}$ relative to P.

Lipschitz Continuity of Set-Valued Mappings

A mapping $S_{feas} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be Lipschitz continuous relative to a (nonempty) set D in $\mathbb{R}^m$ if $D \subset domS$, S is closed-valued on D, and there exists $\kappa \geq 0$ (Lipschitz constant) such that

$$h(S(y'), S(y)) \leq \kappa |y' - y| \text{ for all } y', y \in D,$$

or equivalently, there exists $\kappa \geq 0$ such that

$$S(y') \subset S(y) + \kappa |y' - y| \mathbb{B} \text{ for all } y', y \in D.$$

Distance Characterization of Lipschitz Continuity

Proposition 3C.1 (Distance Characterization of Lipschitz Continuity)

Consider a closed-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a nonempty subset $D \subset domS$. Then S is Lipschitz continuous relative to D with constant κ if and only if

$$d(x, S(y)) \leq \kappa d(y, S^{-1}(x) \cap D) \text{ for all } x \in \mathbb{R}^n \text{ and } y \in D.$$

Proof.

Polyhedral Convex Mappings

Polyhedral Convex Mappings

A mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be polyhedral convex if its graph is a polyhedral convex set.

Theorem 3C.3 (Lipschitz Continuity of Polyhedral Convex Mappings

Any polyhedral convex mapping $S_{feas} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous relative to its domain.

Proof.

Lemma 3C.4 (Hoffman Lemma)

For the set-valued mapping

$$S : y \mapsto \{x \in \mathbb{R}^n | Ax \leq y\} \text{ for } y \in \mathbb{R}^m,$$

where A is a nonzero $m \times n$ matrix, there exists a constant L such that

$$d(x, S(y)) \leq L|(Ax - y)_+| \text{ for every } y \in domS \text{ and every } x \in \mathbb{R}^n.$$

Outer Lipschitz Continuity

A mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be outer Lipschitz continuous at $\bar{y}$ relative to a set D if $\bar{y} \in D \subset domS$, S(y) is a closed set, and there is a constant $\kappa \geq 0$ along with a neighborhood V of $\bar{y}$ such that

$$e(S(y), S(\bar{y})) \leq \kappa |y - \bar{y}| \text{ for all } y \in V \cap D,$$

or equivalently

$$S(y) \subset S(\bar{y}) + \kappa |y - \bar{y}| \mathbb{B} \text{ for all } y \in V \cap D.$$

If S is outer Lipschitz continuous at every point $y \in D$ relative to D with the same κ, then S is said to be outer Lipschitz continuous relative to D.

Outer Lipschitz Continuity

(a) Lipschitz continuous mapping relative to a set D is also outer Lipschitz continuous

(b) Outer Lipschitz continuous at a point y implies outer semicontinuity at y

(c) For single-valued mappings, outer Lipschitz continuity becomes calmness

Polyhedral Mappings

A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ will be called polyhedral if gph S is the union of finitely many sets that are polyhedral convex in $\mathbb{R}^m \times \mathbb{R}^n$

Theorem 3D.1 (Outer Lipschitz Continuity of Polyhedral Mappings)

Any polyhedral mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer Lipschitz continuous at every point of its domain.

Polyhedral Mappings

Theorem 3D.3 (isc Criterion for Lipschitz Continuity)

Consider a set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and a convex set $D \subset domS$ such that $S(y)$ is closed for every $y \in D$. Then S is Lipschitz continuous relative to D with constant κ if and only if S is both isc relative to D and outer Lipschitz continuous relative to D with constant κ.

Polyhedral Mappings

Corollary 3D.4 (Lipschitz Continuity of Polyhedral Mappings)

Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be polyhedral and let $D \subset domS$ be convex. Then S is isc relative to D if and only if S is actually Lipschitz continuous relative to D. Thus, for a polyhedral mapping, continuity relative to its domain implies Lipschitz continuity.

Thank you for your attention!