Inverse and Implicit Mapping Theorem

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- Introduction - Aubin Property, Metric Regularity

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- Inverse and Implicit Mapping Theorem

$X, Y-$ metric spaces, ρ_X, ρ_Y metrics: $\rho_X(\hat{x}, x)$ measures the distance between $\hat{x}$ and x in X
special cases:

1. $X-$ complete metric space (Cauchy sequence convergent)
   $Y-$ linear with a shift-invariant metric ρ:
   $$\rho(\hat{y} + z, y + z) = \rho(\hat{y}, y) \ \forall \hat{y}, y, z \in Y$$
2. $X, Y-$ Banach spaces (complete normed spaces),
   $\rho_X(\hat{x}, x) := \|\hat{x} - x\|_X$
3. $X = R^n$, $Y = R^m$, $\rho(\hat{x}, x) := |\hat{x} - x|$

A set C is said to be *locally closed at* $x \in C$ if there exists a neighborhood U of x such that the intersection $C \cap U$ is closed.

Aubin Property - definition

A mapping $S : Y \rightrightarrows X$ is said to have the *Aubin property at* $\bar{y} \in Y$ *for* $\bar{x} \in X$ if $\bar{x} \in S(\bar{y})$, the graph of S is locally closed at $(\bar{y}, \bar{x})$, and there is a constant $\kappa \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$e(S(\hat{y}) \cap U, S(y)) \leq \kappa\rho(\hat{y}, y) \text{ for all } \hat{y}, y \in V$$

The infimum of κ over all such combinations of κ, U and V is called the *Lipschitz modulus of* S *at* $\bar{y}$ *for* $\bar{x}$ and denoted by $lip(S; \bar{y} \,|\, \bar{x})$. The absence of this property is signaled by $lip(S; \bar{y} \,|\, \bar{x}) = \infty$.

Aubin Property - remarks

- for S single-valued on a neighborhood of $\bar{y}$, $lip(S;\bar{y}\,|\,\bar{x}) = lip(S;\bar{y})$
- in contrast to Lipschitz continuity, the Aubin property is tied to a particular point in the graph of the mapping (picture)
- Theorem 3E.3 (Truncated Lipschitz Continuity Under Convex-Valuedness)
- Theorem 3E.6 (Distance Function Characterization of Aubin Property)

Metric Regularity - definition

A mapping $F : X \rightrightarrows Y$ is said to be *metrically regular at* $\bar{x} \in X$ *for* $\bar{y} \in Y$ if $\bar{y} \in F(\bar{x})$, the graph of F is locally closed at $(\bar{x}, \bar{y})$, and there is a constant $\kappa \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } x, y \in U \times V$$

The infimum of κ over all such combinations of κ, U and V is called the *regularity modulus for* F *at* $\bar{x}$ *for* $\bar{y}$ and denoted by $reg(F; \bar{x} \,|\, \bar{y})$. The absence of this property is signaled by $reg(F; \bar{x} \,|\, \bar{y}) = \infty$.

Equivalence of Metric Regularity and (Inverse) Aubin Property

Theorem (3E.7 / 5A.3)

For Banach spaces X and Y, a mapping $F : X \rightrightarrows Y$ and a constant $\kappa \geq 0$, the following properties with respect to a pair $(\bar{x}, \bar{y}) \in gphF$ are equivalent:

(b) *F is metrically regular at $\bar{x}$ for $\bar{y}$ with constant κ;*

(c) *F^{-1} has the Aubin property at $\bar{y}$ for $\bar{x}$ with constant κ.*

Moreover $reg(F; \bar{x} \,|\, \bar{y}) = lip(F^{-1}; \bar{y} \,|\, \bar{x})$.

Equivalence of Metric Regularity and (Inverse) Aubin Property (and Linear Openness)

Theorem (3E.7 + 3E.9 / 5A.3)

For Banach spaces X and Y, a mapping $F : X \rightrightarrows Y$ and a constant $\kappa \geq 0$, the following properties with respect to a pair $(\bar{x}, \bar{y}) \in gphF$ are equivalent:

(a) *F is linearly open at $\bar{x}$ for $\bar{y}$ with constant κ;*

(b) *F is metrically regular at $\bar{x}$ for $\bar{y}$ with constant κ;*

(c) *F^{-1} has the Aubin property at $\bar{y}$ for $\bar{x}$ with constant κ.*

Moreover $reg(F; \bar{x} \,|\, \bar{y}) = lip(F^{-1}; \bar{y} \,|\, \bar{x})$.

Settings

X complete metric space
Y linear metric space with a shift-invariant metric
P metric space, with all metrics denoted by ρ

$$f : P \times X \to Y, F : X \rightrightarrows Y$$

generalized equation:

$$f(p, x) + F(x) \ni 0$$

and solution mapping:

$$S(p) = \{x \,|\, f(p, x) + F(x) \ni 0\} \text{ with } \bar{x} \in S(\bar{p})$$

Inverse Mapping Theorem with Metric Regularity / Extended Lyusternik - Graves Theorem

Theorem (3F.1 / 5E.1)

Let X, Y, F be as in Settings and consider a point $(\bar{x}, \bar{y}) \in gphF$ and a function $g : X \to Y$ with $\bar{x} \in \operatorname{int}\operatorname{dom} g$. Let κ and μ be nonnegative constants such that

$$\kappa\mu < 1, reg(F; \bar{x} \,|\, \bar{y}) \leq \kappa \text{ and } lip(g; \bar{x}) \leq \mu.$$

Then

$$reg(g + F; \bar{x} \,|\, g(\bar{x}) + \bar{y}) \leq \frac{\kappa}{1 - \kappa\mu}.$$

Implicit Mapping Theorem with Metric Regularity / Extended Lyusternik - Graves Theorem in Implicit Form

Theorem (3F.8 / 5E.5)

Let $X, Y, P, f, F, S, \bar{x}, \bar{p}$ be as in Settings.
Let $h : X \to Y$ be a strict estimator of f with respect to x uniformly in p at $(\bar{p}, \bar{x})$ with constant μ, and suppose that $h + F$ is metrically regular at $\bar{x}$ for 0 with $reg(h + F; \bar{x} \,|\, 0) \leq \kappa$. Assume

$$\kappa\mu < 1 \ and \ \widehat{lip}_p(f; (\bar{p}, \bar{x})) \leq \gamma < \infty.$$

Then S has the Aubin property at $\bar{p}$ for $\bar{x}$, and moreover

$$lip(S; \bar{p} \,|\, \bar{x}) \leq \frac{\kappa\gamma}{1 - \kappa\mu}.$$

Remarks

Theorem 3F.1 (Inverse Mapping Theorem with Metric Regularity) and Theorem 5E.1 (Extended Lyusternik - Graves Theorem) are generalization of

- Theorem 1E.3 (Inverse Function Theorem Beyond Differentiability)
- Theorem 2B.8 (Inverse Function Theorem for Set-Valued Mappings)

Theorem 3F.8 (Implicit Mapping Theorem with Metric Regularity) and Theorem 5E.5 (Extended Lyusternik - Graves Theorem in Implicit Form) are generalization of

- Theorem 1E.13 (Implicit Function Theorem Beyond Differentiability)
- Theorem 2B.7 (Implicit Function Theorem for Generalized Equations)

Contraction Mapping Principle for Set-Valued Mappings

Theorem (5E.2)

Let X be as in Settings and consider a set-valued mapping $\Phi : X \rightrightarrows X$ and a point $\bar{x} \in X$. Suppose that there exist scalars $a > 0$ and $\lambda \in (0,1)$ such that the set $\mathrm{gph}\Phi \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x}))$ is closed and

(a) $d(\bar{x}, \Phi(\bar{x})) < a(1-\lambda)$;

(b) $e(\Phi(u) \cap \mathbb{B}_a(\bar{x}), \Phi(v)) \leq \lambda\rho(u,v)$ *for all* $u, v \in \mathbb{B}_a(\bar{x})$.

Then Φ has a fixed point in $\mathbb{B}_a(\bar{x})$; that is, there exists $x \in \mathbb{B}_a(\bar{x})$ such that $x \in \Phi(x)$.

Comments

- Sketch of the proof of Contraction Mapping Principle on blackboard
- Proof of Extended Lyusternik - Graves Theorem in Implicit Form, using Contraction Mapping Principle on blackboard
- Clearly, Extended Lyusternik - Graves Theorem is the particular case of Extended Lyusternik - Graves Theorem in Implicit Form for $P = Y$, $h = 0$, and $f(p, x) = -p + g(x)$.

Thank you for your attention!