Numerical methods in continuum mechanics 1

Tutorial sheet 7: Pogramming, 28 June 8:30 – 10:00

Tutorial sheet 8: Pogramming, 29 June 8:30 – 10:00

In the following, we solve the Stokes problem

$$\begin{aligned} -\Delta v + \nabla p &= f \text{ in } \Omega \\ \operatorname{div} v &= 0 \text{ in } \Omega \\ v &= 0 \text{ on } \partial\Omega \end{aligned}$$

with $v \in H^1(\Omega)$ and $p \in L^2_0(\Omega)$ using the MINI element for the domain $\Omega = (0,1)^2$.

34. From NuEPDE, you know already the mass matrix $M_h = (\int_\Omega \varphi_i(x)\varphi_j(x)\mathrm{d}x)_{i,j=1}^N$ and the stiffness matrix $K_h = \sum_{k=1}^2 (\int_\Omega \frac{\partial}{\partial x_k}\varphi_i(x)\frac{\partial}{\partial x_k}\varphi_j(x)\mathrm{d}x)_{i,j=1}^N$.
Now, implement for $k = 1,2$ the gradient matrices $D_{h,k} = (\int_\Omega \varphi_i(x)\frac{\partial}{\partial x_k}\varphi_j(x)\mathrm{d}x)_{i,j=1}^N$.

35. Setup of the *classical* Stokes matrix (mixed formulation)

$$\mathcal{A}_h = \left(\begin{array}{cc|c} K_h & 0 & D_{1,h}^T \\ 0 & K_h & D_{2,h}^T \\ \hline D_{1,h} & D_{2,h} & 0 \end{array}\right) =: \left(\begin{array}{c|c} A_h & B_h^T \\ \hline B_h & 0 \end{array}\right)$$

36. Incoroporate the Dirichlet boundary conditions for the velocity v_h. Note that this modifies both, K_h and $D_{k,h}$, the matrix $D_{k,h}$ in a non-symmetric way.

37. Solve the problem $\mathcal{A}_h \underline{x}_h = \underline{f}_h$ with the classical Arrow Hurwitz method, cf. pages 52 and 53 in the lecture notes. How does this work? Determine convergence rates. What can we say about the h-dependence of the convergence rates?

Remark: You can choose $\underline{x}_h = (\underline{v}_{1,h}, \underline{v}_{2,h}, \underline{p}_h)$ to be a vector of random entries and $\underline{f}_h = 0$.

38. Does the approach from exercise 37 lead to $p_h \in L^2_0(\Omega)$? One would expect that this is not the case. So, one has to project p_h into $L^2_0(\Omega)$ in the beginning of the algorithm and after every update of p_h.

For this purpose, one constructs the vector $\underline{q}_h = (\int_\Omega \varphi_i(x)\mathrm{d}x)_{i=1}^N$ in the beginning of the algorithm and compute

$$\underline{p}_h^{(new)} = \underline{p}_h - \frac{\underline{q}_h^T \underline{p}_h}{\underline{q}_h^T \underline{q}_h}\underline{q}_h.$$

Does this improve the situation?

39. Implement the stabilization matrix C_h, which comes from the elimination of the bubble functions (static condensation – cf. page 49 in the lecture notes).

40. Setup of the *improved* Stokes matrix (mixed formulation with MINI element)

$$\mathcal{A}_h = \left(\begin{array}{cc|c} K_h & 0 & D_{1,h}^T \\ 0 & K_h & D_{2,h}^T \\ \hline D_{1,h} & D_{2,h} & -C_h \end{array}\right) =: \left(\begin{array}{c|c} A_h & B_h^T \\ \hline B_h & -C_h \end{array}\right)$$

and solve the problem with the classical Arrow Hurwitz method, cf. pages 52 and 53 in the lecture notes. Does this work now? What can we say about the h-dependence of the convergence rates?

41. Implement the Bramble-Pasciak conjugate gradient method[1] and apply it to the Stokes problem.
The Bramble-Pasciak conjugate gradient method is a *preconditioned* conjugate gradient method.

The *preconditioned* conjugate gradient method for a general matrix problem $M\underline{x} = \underline{b}$ and preconditioner $\widehat{M}^{-1}$ reads as follows:

$$
\begin{aligned}
&\underline{r}^{(0)} = \underline{b} - M\underline{x}^{(0)} \\
&\underline{h}^{(0)} = \widehat{M}^{-1}\underline{r}^{(0)} \\
&\underline{d}^{(0)} = \underline{h}^{(0)} \\
&\text{for } k = 0, 1, 2, 3, \ldots \\
&\qquad \underline{z}^{(k)} = M\underline{d}^{(k)} \\
&\qquad \alpha^{(k)} = \frac{\underline{r}^{(k)} \cdot \underline{h}^{(k)}}{\underline{d}^{(k)} \cdot \underline{z}^{(k)}} \\
&\qquad \underline{x}^{(k+1)} = \underline{x}^{(k)} + \alpha^{(k)}\underline{d}^{(k)} \\
&\qquad \underline{r}^{(k+1)} = \underline{r}^{(k)} - \alpha^{(k)}\underline{z}^{(k)} \\
&\qquad \underline{h}^{(k+1)} = \widehat{M}^{-1}\underline{r}^{(k)} \\
&\qquad \beta^{(k)} = \frac{\underline{r}^{(k+1)} \cdot \underline{h}^{(k+1)}}{\underline{r}^{(k)} \cdot \underline{h}^{(k)}} \\
&\qquad \underline{d}^{(k+1)} = \underline{h}^{(k+1)} + \beta^{(k)}\underline{d}^{(k)} \\
&\text{next } k
\end{aligned}
$$

For Bramble-Pasciak conjugate gradient method, we choose

$$
\begin{aligned}
&M := \widehat{M}\widehat{\mathcal{A}}_h^{-1}\mathcal{A}_h \\
&\underline{b} := \widehat{\mathcal{A}}_h^{-1}\underline{f}_h \\
&\underline{x}^{(0)} := \text{random vector} \\
&\widehat{M} := \begin{pmatrix} A_h - \kappa I & \\ & I \end{pmatrix}, \text{ where } A_h \text{ is as in exercise 40} \\
&\widehat{\mathcal{A}}_h^{-1} := \begin{pmatrix} \kappa^{-1}I & 0 \\ \kappa^{-1}B & -I \end{pmatrix}
\end{aligned}
$$

with κ as large as possible such that $A_h - \kappa I$ is positive definte. Note that

$$
\widehat{M}^{-1}M = \widehat{\mathcal{A}}_h^{-1}\mathcal{A}_h = \left(\begin{array}{c|c} \kappa^{-1}A_h & \kappa^{-1}B_h^T \\ \hline \kappa^{-1}B_h(A_h - \kappa I) & C_h + \kappa^{-1}B_hB_h^T \end{array}\right)
$$

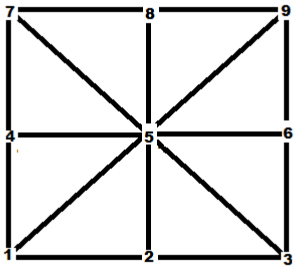
which is self-adjoined in the scalar product $(\cdot,\cdot)_{\widehat{M}}$.

What can we say about the h-dependence of the convergence rates? Compare this with the Uzawa solver.

Hints for all exercises:

Domain: Everything can be done for $\Omega = (0,1)^2$. The grid level $\ell = 1$ is obtained as follows:

[1] as introduced in James H. Bramble and Joseph E. Pasciak: *A Preconditioning Technique for Indefinite Systems Resulting from Mixed Approximations of Elliptic Problems*
http://www.ams.org/journals/mcom/1988-50-181/S0025-5718-1988-0917816-8/S0025-5718-1988-0917816-8.pdf



The next grid levels $\ell = 2, 3, 4$ are obtained by uniform refinement (one triangle is subdivided into 4 congruent subtriangles).

Presentation: Please write down step-by-step what your algorithm / method is doing such that you can present the main steps on the black board.

Implement the method with the code which you have been working on in the NuEPDE exercises.

Present the results of the algorithms. For the matrix-setup problems 34, 36 and 39, please print the entries of the matrices for $\ell = 1$. For the solving problems 37, 38, 40 and 41, please present the number of iterations needed to reduce the residual $\|\underline{f}_h - \mathcal{A}_h \underline{x}_h^{(n)}\|_{K_h \times K_h \times M_h}$ to $\epsilon = 10^{-6}$ times the initial residual $\|\underline{f}_h - \mathcal{A}_h \underline{x}_h^{(0)}\|_{K_h \times K_h \times M_h}$ for several grid levels $\ell = 1, 2, 3$ and, as far as possible, for larger values of ℓ.