Numerical methods in continuum mechanics 1

Tutorial sheet 6: Thu 28 05 2015

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28. Let $\Omega = (0,1)^3$ be the unit cube. The domain Ω is subdiveded into two (open) subdomains T_1 and T_2 by some plane. Assume that there is some function $v : \Omega \to \mathbb{R}$ is is piecewise continuously differentiable, i.e., for $i = 1$ and $i = 2$

- $v|_{T_i}$ is continuously differentiable and
- there is some continuously differentiable function $v_i : \overline{T}_i \to \mathbb{R}$ such that $v_i|_{T_i} = v|_{T_i}$, where $\overline{T}_i$ is the closure of T_i (extension to the closure).

Show that $v \in H^1(\Omega)$ if and only if $v_1 n_1 + v_2 n_2 = 0$ on $\overline{T}_i$, where n_i is the outer normal vector. Show that this is equivalent to $v \in C^1(\Omega)$.

29. Let $\Omega = (0,1)^3$ be the unit cube. The domain Ω is subdiveded into two (open) subdomains T_1 and T_2 by some plane. Assume that there is some function $v : \Omega \to \mathbb{R}^3$ is is piecewise continuously differentiable, i.e., for $i = 1$ and $i = 2$

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Show that $v \in H(\mathrm{div}, \Omega)$ if and only if $v_1 n_1 + v_2 n_2 = 0$ on $\overline{T}_i$, where n_i is the outer normal vector.

30. Let $\Omega = (0,1)^3$ be the unit cube.

Let $\mathrm{curl}\ u := \nabla \times u = \begin{pmatrix} -\partial_3 v_2 + \partial_2 v_3 \\ \partial_3 v_1 - \partial_1 v_3 \\ -\partial_2 v_1 + \partial_1 v_2 \end{pmatrix}$.

Let $H(\mathrm{curl}, \Omega)$ be the space of all functions $w \in [L^2(\Omega)]^3$ with $\mathrm{curl}\ w \in L^2(\Omega)$ with norm $\|w\|^2_{H(\mathrm{curl},\Omega)} := \|w\|^2_{L^2(\Omega)} + \|\mathrm{curl}\ w\|^2_{L^2(\Omega)}$

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- $v|_{T_i}$ is continuously differentiable and
- there is some continuously differentiable function $v_i : \overline{T}_i \to \mathbb{R}$ such that $v_i|_{T_i} = v|_{T_i}$, where $\overline{T}_i$ is the closure of T_i (extension to the closure).

Show that $v \in H(\mathrm{curl}, \Omega)$ if and only if $v_1 n_1 \times v_2 n_2 = 0$ on $\overline{T}_i$, where n_i is the outer normal vector.

31. Let $M = (-1,1) \times (-1,1)$ be a macro-element with 9 knodes $x_1 = (-1,-1)$, $x_2 = (0,-1)$, $x_3 = (1,-1)$, $x_4 = (-1,-1)$, $x_5 = (0,0)$, $x_6 = (1,0), x_7 = (-1,1)$, $x_8 = (0,1)$, $x_9 = (1,1)$. The macro element consists of 4 subelements $T_1 = (-1,0) \times (-1,0)$, $T_2 = (0,1) \times (-1,0)$, $T_3 = (-1,0) \times (0,1)$, $T_4 = (0,1) \times (0,1)$.

The edges of M are denoted in the following way: $S_1 = [x_1, x_3]$, $S_2 = [x_3, x_9]$, $S_3 = [x_9, x_7]$, $S_4 = [x_7, x_1]$.

Step 1: Make a picture.

Consider the following finite element space:

$$V_h := \{ v_h \in C(\overline{M}) \ : \ v|_{T_i} \in Q_1 \text{ for all } i \in \{1,2,3,4\} \ \},$$

where Q_1 is the set of binlinear functions.

Now, construct for a given function $u \in C(\overline{M})$ a function $\Pi_h v \in V_h$ such that

$$[\Pi_h v](x_i) = \frac{1}{area(\Delta_i)} \int_{\Delta_i} v \mathrm{d}x \qquad \text{for all } i \in \{1,3,5,7,9\},$$

where Δ_i is the union of all elements T_j with $x_i \in \overline{T}_j$ AND

$$\int_{S_i} v \mathrm{d}x = \int_{S_i} [\Pi_h v] \mathrm{d}x \qquad \text{for all } i \in \{1,2,3,4\},$$

Hint: Set up the nodal basis of V_h: $(\varphi_1, \ldots, \varphi_9)$. Then every $v_h in V_h$ can be expressed as linear combination of them:

$$v_h = \sum_{i=1}^{9} \alpha_i \varphi_i.$$

Determine the α_i's.

32. Show under the notations and assumptions of Ex. 31 that there exists a constant $c > 0$ with

$$\|\Pi_h v\|_{H^1(\Omega)} \le c\|v\|_{H^1(\Omega)}. \tag{1}$$

Hint: Use that the α_i's can be represented using the terms $\int_{\Delta_i} v \mathrm{d}x \le c\|v\|_{H^1(\Omega)}$ and $\int_{S_i} v \mathrm{d}x$. Show and use $\int_{S_i} v \mathrm{d}x \le c\|v\|_{H^1(\Omega)}$ using some trace theorem.

33. Show under the notations and assumptions of Ex. 31 that there exists a constant $c > 0$ with

$$\|v - \Pi_h v\|_{H^1(\Omega)} \le c|v|_{H^1(\Omega)}.$$

Hint: Use $\|v - \Pi_h v\|_{H^1(\Omega)} = \|(v + c) - \Pi_h(v + c)\|_{H^1(\Omega)}$, where c is a constant. Then use the triangular inequality, inequality (1) and Poincare inequality.