Numerical methods in continuum mechanics 1

Tutorial sheet 5: Thu 21 05 2015

Remark: Each of the exercises can be completed even if one of the preceeding exercises has not been completed!

21. Let $\Omega \subseteq \mathbb{R}^d$ be an open and bounded set with Lipschitz-continuous boundary $\partial\Omega$. Show that for all $v \in C^1(\bar{\Omega}, \mathbb{R}^d)$

$$\int_\Omega q \operatorname{div} v \, \mathrm{d}x = \int_{\partial\Omega} v \cdot n \, q \, \mathrm{d}s - \int_\Omega v \cdot \operatorname{grad} q \, \mathrm{d}x$$

holds for all $q \in C^1(\bar{\Omega})$.

Now, show that the operator $\gamma_n : C^1(\bar{\Omega}, \mathbb{R}^d) \to H^*$, given by

$$\langle \gamma_n v, q|_{\partial\Omega} \rangle = \int_{\partial\Omega} v \cdot n \, q \, \mathrm{d}s$$

with $H = \{q|_{\partial\Omega} \ : \ q \in C^1\}$,

$$H^* = \{l \ : \ l \text{ bounded linear functional } H \to \mathbb{R}\}$$

with standard dual norm

$$\|l\|_{H^*} = \sup_{q \in C^1} \frac{\langle l, q \rangle}{\|q\|_1},$$

is linear and bounded:

$$\|\gamma_n v\|_{H^*} \leq c \|v\|_{H(div,\Omega)}.$$

This shows that there is a continuous extension of the operator γ_n to $H(div, \Omega) \to H^{-1/2}(\partial\Omega)$, where $H^{1/2}(\partial\Omega) = \{\gamma q \ : \ q \in H^1(\Omega)\}$ and $H^{-1/2}(\partial\Omega) = [H^{1/2}(\partial\Omega)]^*$ and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega$ is the (standard) trace operator.

22. Let V and P be Hilbert spaces. Let $A : V \to V^*$, $B : P \to V^*$ and $C : P \to P^*$ be bounded linear operators. Note that $B^* : V \to P^*$. Let

$$\mathcal{A} = \begin{pmatrix} A & B^* \\ B & -C \end{pmatrix}, \qquad x = \begin{pmatrix} v \\ p \end{pmatrix}$$

and $\|x\|_X^2 = \|v\|_V^2 + \|p\|_P^2$. Assume that

$$\underline{c}_X \|x\|_X \leq \|\mathcal{A}x\|_{X^*} \leq \bar{c}_X \|x\|_X$$

for all $x \in X = V \times P$. Show that

$$\underline{c}_X^2 \|v\|_V^2 \leq \|Av\|_{V^*}^2 + \|B^* v\|_{P^*}^2 \leq \bar{c}_X^2 \|v\|_V^2$$

and

$$\underline{c}_X^2 \|p\|_P^2 \leq \|Bp\|_{V^*}^2 + \|Cp\|_{P^*}^2 \leq \bar{c}_X^2 \|p\|_P^2$$

for all $v \in V$ and all $p \in P$.

Hint: Choose $x = (v, 0)$ and $x = (0, p)$.

23. Having the notations of the last exercise. Now, assume that

$$\underline{c}_V^2 \|v\|_V^2 \leq \|Av\|_{V^*}^2 + \|B^* v\|_{P^*}^2 \leq \bar{c}_V^2 \|v\|_V^2 \tag{1}$$

and

$$\underline{c}_P^2 \|p\|_P^2 \leq \|Bp\|_{V^*}^2 + \|Cp\|_{P^*}^2 \leq \bar{c}_P^2 \|p\|_P^2 \tag{2}$$

for all $v \in V$ and all $p \in P$. Show that there is a constant $\overline{c}_X$ such that

$$\|\mathcal{A}x\|_{X^*} \leq \overline{c}_X \|x\|_X$$

for all $x \in X$. How should $\overline{c}_X$ be chosen?

24. Having the assumptions of exercise 23. Show that

$$\|\mathcal{A}x\|_{X^*} \geq (\eta - \xi)\|x\|_X, \tag{3}$$

where

$$\xi = \frac{\sqrt{\|Av\|^2_{V^*} + \|Cp\|^2_{P^*}}}{\|(v,p)\|_X} \qquad \text{and} \qquad \eta = \frac{\sqrt{\|B^*v\|^2_{P^*} + \|Bp\|^2_{V^*}}}{\|(v,p)\|_X}.$$

25. Having the assumptions of exercise 23. Assume that the operators A and C are symmetric and coercive, i.e., $\langle Au, v\rangle = a(u,v) = a(v,u) = \langle Av, u\rangle$ and $a(u,u) \geq 0$ (and the same for C). Show that

$$a(v,v) + c(p,p) \geq \frac{\sqrt{2}}{\overline{c}_X}\xi^2\|(v,p)\|_X$$

and consequencly

$$\|\mathcal{A}x\|_{X^*} \geq \frac{\sqrt{2}}{\overline{c}_X}\xi^2\|x\|_X, \tag{4}$$

where ξ is defined as in exercise 24.

26. Having the assumptions of exercise 23. Assume that the operators A and C are symmetric and coercive. Show that

$$\xi^2 + \eta^2 \geq \min\{\underline{c}_V^2, \underline{c}_P^2\},$$

where ξ and η are defined as in exercise 24. Use (3) and (4) and observee

$$\|\mathcal{A}x\|_{X^*} \geq \max\{\eta - \xi, \frac{\sqrt{2}}{\overline{c}_X}\xi^2\}\|x\|_X.$$

Use these two results to show that there is a constant $\underline{c}_X$, only depending on $\overline{c}_X$, $\underline{c}_V$ and $\underline{c}_P$ such that

$$\|\mathcal{A}x\|_{X^*} \geq \underline{c}_X\|x\|_X.$$

27. Use the theorem that was shown in the exercises 23 - 26, i.e., that

$$\underline{c}_X\|x\|_X \leq \|\mathcal{A}x\|_{X^*} \leq \overline{c}_X\|x\|_X$$

if (1) and (2) is satisfied.

Use this theorem to show existence and uniqueness of the problem in Section 1.2.1 (Incompressible and Almost Incompressible Materials) for the general case $\lambda < \infty$ having the function spaces V and $P = Q$ as given in that section.