Numerical methods in continuum mechanics 1

Tutorial sheet 4: Thu 30 04 2015

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15 (Gradient matrix). Let $\Omega \subseteq \mathbb{R}^2$ with a triangular subdivision of Ω into $\{\Omega_i\}$.

Define the gradient matrix $D := ((\varphi^{(j)}, \nabla\psi^{(k)})_{L^2(\Omega)})_{k=1,\dots,n;l=1,\dots,m}$, with $V = \operatorname{span}\{\varphi^{(j)}\} \subseteq H^1(\Omega)$ and $P = \operatorname{span}\{\psi^{(k)}\} \subseteq [L^2(\Omega)]^2$. So, D is representing the off-diagonal parts of the discretization of the Stokes problem.

Let K be the standard stiffness matrix on V and M_p the standard mass matrix on P.

Show:

- $DM_p^{-1}D^T = K$ if V is the Courant element (piecewise linear, globally continuous) and P is piecewiese constant.
- $DM_p^{-1}D^T \neq K$ if both, V and P are the Courant element (find a counter example).

Hint for the second statement: Use $\Omega = (0,1)^2$ and subdivide it into two triangles.

16 (discrete Babuska-Aziz). Assume the notations and conditions of the theorem of Babuska-Aziz and let $F \in Y^*$. Then the variational problem

$$\text{find } x \in X \text{ such that} \qquad \mathrm{a}(x,y) = \langle F, y\rangle \qquad \text{for all } y \in Y.$$

has a unique solution. Let $X_h \subseteq X$ and $Y_h \subseteq Y$ be finite-dimensional subspaces. Assume that

- (2') There is a constant $\tilde{\mu}_1 > 0$ such that $\inf_{x_h\in X_h} \sup_{y_h\in Y_h} \frac{\mathrm{a}(x_h,y_h)}{\|x_h\|_X\|y_h\|_Y} \geq \tilde{\mu}_1 > 0$.
- (3') For each $y_h \in Y_h\backslash\{0\}$, there is a $x_h \in X_h$ such that $\mathrm{a}(x_h, y_h) \neq 0$.

Show that the variational problem

$$\text{find } x_h \in_h X \text{ s.t.} \qquad \mathrm{a}(x_h, y_h) = \langle F, y_h\rangle \qquad \text{for all } y_h \in Y_h.$$

has a unique solution.

17 (Error estimate). Show that the following estimate is satisfied for the discretization error in 16.:

$$\|x - x_h\|_X \leq (1 + \frac{\mu_2}{\tilde{\mu}_1}) \inf_{w_h\in X_h} \|x - w_h\|_X.$$

Hint: Use $\|x-x_h\|_X \leq \|x-w_h\|_X + \|w_h - x_h\|_X$. Show and use $\mathrm{a}(x_h - w_h, y_h) = \mathrm{a}(x - w_h, y_h)$.

18 (Weak gradient). Let $\Omega = (0,1)$. Show that

$$\|\operatorname{grad} p\|_{H^{-1}(\Omega)} = \|p\|_{L^2(\Omega)}$$

for all $p \in L^2_0(\Omega)$.

Here, $\operatorname{grad} p \in H^{-1}(\Omega) = [H^1_0(\Omega)]^*$ is given by $\langle \operatorname{grad} p, q\rangle = -\int_0^1 pq' \mathrm{d}x$. The norm is given by $\|\cdot\|_{H^{-1}(\Omega)} = \|\cdot\|_{[H^1_0(\Omega)]^*} = \sup_{q\in H^1_0(\Omega)} \frac{\langle\cdot,q\rangle}{|q|_{H^1_0(\Omega)}}$

Hint: You may assume that p is a smooth function.

19 (Weak divergence). Let $\Omega \in \mathbb{R}^d$ be an open and bounded set. Let $u \in [L^2(\Omega)]^d$. If a function $w \in L^2(\Omega)$ exists such that

$$\int_\Omega w\, v \,\mathrm{d}x = -\int_\Omega u \cdot \operatorname{grad} v \,\mathrm{d}x$$

for all $v \in H^1(\Omega)$, then we call w the weak divergence of v and we write $w = \operatorname{div} u$. One defines now

$$H(\operatorname{div}, \Omega) = \{v \in [L^2(\Omega)]^d \ : \ \operatorname{div} v \in L^2(\Omega)\}.$$

This function space with the scalar product $(u, v)_{H(\operatorname{div},\Omega)} := (u, v)_{L^2(\Omega)} + (\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega)}$ is a Hilbert space.

Let $f \in L^2(\Omega)$. Using Lax-Milgram and Friedrichs' inequality, we obtain that there is a unique solution $u \in H^1_0(\Omega)$ of the problem

$$\text{find } u \in H^1_0(\Omega) \text{ such that} \qquad \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x \qquad \text{for all } v \in H^1_0(\Omega)$$

Since $u \in H^1_0(\Omega)$, we have $\operatorname{grad} u \in L^2(\Omega)$.
Show:

- $\operatorname{grad} u \in H(\operatorname{div}, \Omega)$
- $\operatorname{div} \operatorname{grad} u = -f$

20 (Inf sup condition involving the weak divergence). Let $\Omega \in \mathbb{R}^d$ be an open and bounded set.

Show that there is a constant $\beta_1 > 0$ such that

$$\inf_{q \in L^2(\Omega)} \sup_{v \in H(\operatorname{div},\Omega)} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|v\|_{H(\operatorname{div},\Omega)}} \geq \beta_1.$$

Hint: Let q be abitrarily but fixed. Choose $v = -\operatorname{grad} w$, where $w \in H^1_0(\Omega)$ solves

$$\text{find } w \in H^1_0(\Omega) \text{ such that} \qquad (\operatorname{grad} w, \operatorname{grad} \tilde{w})_{L^2(\Omega)} = (q, \tilde{w})_{L^2(\Omega)} \qquad \text{for all } \tilde{w} \in H^1_0(\Omega)$$

and observe that $\operatorname{div} v = q$. Use what you (should) have shown in exercise 19.