Numerical methods in continuum mechanics 1

Tutorial sheet 3: Thu 16 04 2015

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10 (Cylindric domain). Let $\Omega = \omega \times (-d, d) \subseteq \Omega^3$ with $d > 0$. Consider the problem:

$$\begin{aligned} -\mathrm{div}\sigma &= f && \text{in } \Omega \\ \sigma &= \lambda \mathrm{tr}\epsilon(u) I + 2\mu\epsilon(u) && \text{in } \Omega \\ u_n = \sigma_T &= 0 && \text{on } \omega \times \{-d, d\} \\ u &= u_D && \text{on } \gamma_D \times (-d, d) \\ \sigma n &= t_N && \text{on } \gamma_N \times (-d, d), \end{aligned}$$

where $\gamma_D \cup \gamma_N = \partial\omega$ and

$$v_n := v \cdot n, \qquad v_t := v - v_n n, \qquad \sigma_n := \sigma n \cdot n, \qquad \sigma_t := \sigma n - \sigma_n n.$$

a. This can be rewritten as variational formulation: Find $u \in V_g$ such that

$$a(u, v) = \langle F, v \rangle \text{ for all } v \in V_0. \tag{1}$$

Determine the spaces V_0, V_g, the bilinear form a and the linear functional F.

Hint: Show and use $\sigma n \cdot v = \sigma_n v_n + \sigma_t \cdot v_t$.

b. Assume that

- $f(x_1, x_2, x_3) = (f_1(x_1, x_2), f_2(x_1, x_2), 0)$,
- $u_D(x_1, x_2, x_3) = (u_{D,1}(x_1, x_2), u_{D,2}(x_1, x_2), 0)$ and
- $t_N(x_1, x_2, x_3) = (t_{N,1}(x_1, x_2), t_{N,2}(x_1, x_2), 0)$.

Then it is reasonable to assume that also the solution u is essentialy 2 dimensional, i.e.: $u(x_1, x_2, x_3) = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$.

Derive a variational equation: Find $\tilde{u} \in \tilde{V}_g$ such that

$$\tilde{a}(\tilde{u}, \tilde{v}) = \langle \tilde{F}, \tilde{v} \rangle, \text{ for all } \tilde{v} \in \tilde{V}_0, \tag{2}$$

where $\tilde{u}(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ and $\tilde{v}(x_1, x_2) = (v_1(x_1, x_2), v_2(x_1, x_2))$. Determine the spaces $\tilde{V}_0$, $\tilde{V}_g$, the bilinear form $\tilde{a}$ and the linear functional $\tilde{F}$.

11. Show that, if $\tilde{u}(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ solves (2), then $u(x_1, x_2, x_3) = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$ solves (1).

Hint: Show that each $v \in V_0$ can be expressed as

$$v(x_1, x_2, x_3) = (v_1(x_1, x_2), v_2(x_1, x_2), 0) + w,$$

where $\int_{-d}^{d} w_1(x)\, \mathrm{d}x_3 = 0$ and $\int_{-d}^{d} w_2(x)\, \mathrm{d}x_3 = 0$. Then show that $a(u, w) = 0$ and $\langle F, w \rangle = 0$.

12 (Closed range theorem for finite dimensional space). Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Show that

$$Ax = f$$

has a solution if and only if $y^T f = 0$ for all $y \in Z^\perp := \{y \;:\; A^T y = 0\}$.

13 (Well posedness of mass matrix). Let $\Omega \subseteq \mathbb{R}^2$ with a triangular subdivision of Ω into $\{\Omega_i\}$. Let $M_i = ((\varphi^{(k)}, \varphi^{(l)})_{L^2(\Omega_i)})_{k,l=1}^n$ be the mass matrices on the elements Ω_i and let $M = ((\varphi^{(k)}, \varphi^{(l)})_{L^2(\Omega)})_{k,l=1}^n = \sum_{i=1}^N M_i$ be the mass matrix on Ω. Show that

$$\frac{\lambda_{max}(M)}{\lambda_{min}(M)} \leq C_1 \max_{i=1\ldots N} \frac{\lambda_{max}(M_i)}{\lambda_{min}(M_i)} \leq C_2,$$

where the constants C_1 and C_2 are independent of the grid size.

Discuss: Which conditions do you need? How do C_1 and C_2 look like?

Hint: Use the Rayleigh-quotients $\lambda_{max}(M) = \sup_x \frac{x^T M x}{x^T x}$ and $\lambda_{min}(M) = \inf_x \frac{x^T M x}{x^T x}$ and the fact that each node only contributes to finitely many elements.

14 (h dependence of the stiffness matrix). Let $\Omega \subseteq \mathbb{R}^2$ with a triangular subdivision of Ω into $\{\Omega_i\}$. Let $K_i = ((\varphi^{(k)}, \varphi^{(l)})_{H^1(\Omega_i)})_{k,l=1}^n = M_i + ((\nabla\varphi^{(k)}, \nabla\varphi^{(l)})_{L^2(\Omega_i)})_{k,l=1}^n$ be the stiffness matrices on the elements Ω_i and let the matrix $K = ((\varphi^{(k)}, \varphi^{(l)})_{H^1(\Omega)})_{k,l=1}^n = M + ((\nabla\varphi^{(k)}, \nabla\varphi^{(l)})_{L^2(\Omega)})_{k,l=1}^n = \sum_{i=1}^N K_i$ be the stiffness matrix on Ω. Show that

$$\frac{\lambda_{max}(K)}{\lambda_{min}(K)} \leq C_1 \max_{i=1\ldots N} \frac{\lambda_{max}(K_i)}{\lambda_{min}(K_i)} \leq C_2 h^{-2},$$

where the constants C_1 and C_2 are independent of the grid size h.

Discuss: Which conditions do you need? How do C_1 and C_2 look like?

15 (Gradient matrix). Let $\Omega \subseteq \mathbb{R}^2$ with a triangular subdivision of Ω into $\{\Omega_i\}$.

Define the gradient matrix $D := ((\varphi^{(j)}, \nabla\psi^{(k)})_{L^2(\Omega)})_{k=1,\ldots,n;l=1,\ldots,m}$, with $V = \text{span}\{\varphi^{(j)}\} \subseteq H^1(\Omega)$ and $P = \text{span}\{\psi^{(k)}\} \subseteq [L^2(\Omega)]^2$. So, D is representing the off-diagonal parts of the discretization of the Stokes problem.

Let K be the standard stiffness matrix on V and M_p the standard mass matrix on P.

Show:

- $D{M_p}^{-1}D^T = K$ if V is the Courant element (piecewise linear, globally continuous) and P is piecewiese constant.
- $D{M_p}^{-1}D^T \neq K$ if both, V and P are the Courant element (find a counter example).

Hint for the second statement: Use $\Omega = (0, 1)^2$ and subdivide it into two triangles.