Numerical methods in continuum mechanics 1

Tutorial sheet 2: Thu 26 03 2015

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4. Show that the Navier Stokes equations in convective form,

$$\frac{\partial v}{\partial t} - \Delta v + v \cdot \nabla v + \nabla p = f$$
$$\text{div } v = 0,$$

are equivalent to the Navier Stokes equations in the conservative form

$$\frac{\partial v}{\partial t} - \Delta v + \text{div } (vv^T) + \nabla p = f$$
$$\text{div } v = 0,$$

or in rotation form

$$\frac{\partial v}{\partial t} - \Delta v + (\nabla \times v) \times v + \nabla P = f$$
$$\text{div } v = 0,$$

with $P = p + \frac{1}{2}v^2$.

5. Let $\Omega := (0, L) \times (0, H)$ for $L > H > 0$ be a rectangular domain. Compute a lower bound for

$$\sup_{v \in H^1(\Omega)^2} \frac{|v|_1^2}{\|\epsilon(v)\|_0^2},$$

by choosing $v(x, y) = (2xy, -x^2)$. Is this $v \in H^1(\Omega)^2$? Why does this lower bound cause problems?

6. Have $A : B = \sum_{i,j} a_{i,j} b_{i,j}$ and $\text{tr}(A) = \sum_i a_{i,i}$ and $\text{dev}(A) = A - \frac{1}{n}\text{tr}(A)I$. Show that

$$A : B = \text{tr}(A^T B),$$

$$\text{tr}(ABA^{-1}) = \text{tr}(B)$$

and

$$\text{tr}(\text{dev}A) = 0$$

holds for all matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$.

7. Consider the linearized elasticity problem: Find $u \in [H^1(\Omega)]^3$ such that

$$\int_\Omega C\epsilon(u) : \epsilon(v) \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x + \int_\Gamma t_N \cdot v \mathrm{d}s \text{ for all } v \in [H^1(\Omega)]^3 \quad (1)$$

Assume that there is a solution $u \in [H^1(\Omega)]^3$. Show that the right-hand side must satisfy the following conditions:

$$\int_\Omega f \mathrm{d}x + \int_\Gamma t_N \mathrm{d}s = 0 \qquad \text{and} \qquad \int_\Omega x \times f \mathrm{d}x + \int_\Gamma x \times t_N \mathrm{d}s = 0. \quad (2)$$

8. Show that each function $v \in [H^1(\Omega)]^3$ can be rewritten as

$$v = \hat{v} + v_0,$$

where $\hat{v} \in \hat{H}(\Omega) := \{v \in H^1(\Omega)^3 : \int_\Omega v \text{ d}x = 0, \int_\Omega \text{curl } v \text{ d}x = 0\}$ and $v_0 \in RM$ (cf. Lemma 2.4 and Remark 2 on page 15).

9. You know that the problem, find $u \in \hat{H}(\Omega)$ such that

$$\int_\Omega C\epsilon(u) : \epsilon(v) \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x + \int_\Gamma t_N \cdot v \mathrm{d}s \text{ for all } v \in \hat{H}(\Omega),$$

has a unique solution.

Show that, assuming (2), the solution of this problem, say u^*, also satisfies (1). Show moreover that (1) is also satisfeid for $u^* + u_0$, no matter how $u_0 \in RM$ is chosen.