Numerical methods in continuum mechanics 1

Tutorial sheet 1: Thu 19 03 2015

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0. Recall what you have learned in Numerical methods for partial differential equations: Assume that $F \in (L^2(\Omega))^*$ is given. Then the minimization problem

$$\min_{u \in H_0^1(\Omega)} \frac{1}{2} \int_\Omega \text{grad } u \cdot \text{grad } u \text{ d}x - \langle F, u \rangle$$

is equivalent to the variational formulation of the Poisson equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \text{grad } u \cdot \text{grad } v \text{ d}x = \langle F, v \rangle$$

for all test functions $v \in H_0^1(\Omega)$. This is discussed in W. Zulehner's lecture notes[1] on page 8.

1. Assume that $F \in (L^2(\Omega))^*$ is given. Show that the minimization problem (A)

$$\min_{u \in H_0^1(\Omega), \text{div } u = 0} \underbrace{\frac{1}{2} \int_\Omega \text{grad } u : \text{grad } u \text{ d}x - \langle F, u \rangle}_{J(u) :=}$$

is equivalent to the following problem: (B) Find $u \in H_0^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\mathcal{L}(u, q) \leq \mathcal{L}(u, p) \leq \mathcal{L}(w, p)$$

for all $w \in H_0^1(\Omega)$ and $q \in L^2(\Omega)$, where $\mathcal{L}$ is given by:

$$\mathcal{L}(w, q) := L(w) + \int_\Omega q \text{ div } w \text{ d}x.$$

2. Show that the problems (A) and (B) are equivalent to the Stokes problem (1.4) in the lecture notes.

3. Assume $\Omega \subseteq \mathbb{R}^2$. Define

$$u := \text{curl } \psi := \begin{pmatrix} -\partial\psi/\partial y \\ \partial\psi/\partial x \end{pmatrix}.$$

Show that u is a solution of the Stokes problem (1.1), (1.2) if and only if

$$-\mu \Delta^2 \psi = \text{curl } f.$$

How is curl f defined?

[1] http://www.numa.uni-linz.ac.at/Teaching/LVA/2008w/NuPDE/numpde_en.pdf