# T U T O R I A L

## "Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

**Tutorial 12** Tuesday, 24 June 2014, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 120.

### 3.8 A posteriori Discretization Error Analysis

**41** Let $\Omega = (0, 1)$, and let us consider an equidistant subdivision into finite elements $[x_{i-1}, x_i] = [(i-1)h, i\,h]$, $i = 1, \ldots, n$. For each node $x_i = i\,h$, $i = 1, \ldots, n-1$, we define the local $L_2$-projection $P_i : L_2(x_{i-1}, x_{i+1}) \to \mathcal{P}_0(x_{i-1}, x_{i+1}) = \mathbb{R}$ by

$$\int_{x_{i-1}}^{x_{i+1}} (P_i v)\, q\, dx \;=\; \int_{x_{i-1}}^{x_{i+1}} v\, q\, dx \qquad \forall q \in \mathcal{P}_0(x_{i-1}, x_{i+1}) \quad \forall v \in L_2(x_{i-1}, x_{i+1})\,,$$

where $\mathcal{P}_0(x_{i-1}, x_{i+1})$ are the constant functions on $(x_{i-1}, x_{i+1})$. Show that

1) $P_i v = \frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}} v(x)\, dx$,
2) $\|v - P_i v\|_{L_2(x_{i-1}, x_{i+1})} \le c\, h\, \|v'\|_{L_2(x_{i-1}, x_{i+1})} \qquad \forall v \in H^1(x_{i-1}, x_{i+1})$.

**42** Let $V_0 := H^1_0(0, 1)$ and $V_{0h} := \operatorname{span}\{p^{(j)} : j = 1, \ldots, n-1\}$, where $p^{(j)}$ is the nodal basis function associated to the node $x_j$. We define Clément's interpolator $I_h : L_2(0, 1) \to V_{0h} \subset V_0$ by

$$(I_h u)(x) := \sum_{j=1}^{n-1} (P_j u)\, p^{(j)}(x) \qquad \text{for } x \in [0, 1]\,.$$

Show that $\|u - I_h u\|_{L_2(0,1)} \;\le\; c\, h\, \|u'\|_{L_2(0,1)} \quad \forall u \in V_0\,.$
*Hint:* Follow your lecture notes. The difference here is that we have to treat homogeneous Dirichlet boundary conditions ! Show and use the Friedrichs inequality

$$\|u\|_{L_2(x_0, x_1)} \;\le\; c_F\, h\, |u|_{H^1(x_0, x_1)}\,,$$

with $c_F \ne c_F(h)$. The same inequality is valid for the interval $(x_{n-1}, x_n)$.

**43*** Show that $|u - I_h u|_{H^1(0,1)} \;\le\; c\, \|u\|_{H^1(0,1)}\,.$

**44** In Section 3.6.2 of our lecture, we derived the residual error estimator for the Dirichlet problem for Poisson's equation. How do we have to modify this estimator such that it works for our model problem CHIP ? Derive the right residual error estimator for the CHIP problem !