T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 09 Tuesday, 27 May 2014, Time: $10^{\underline{15}} - 11^{\underline{45}}$, Room: S2 120.

3.6 Discretization Error Analysis

33 Show that, for $d = 1$: $\Omega = (0,1)$; $k = 1$: $\mathcal{F}(\Delta) = \mathcal{P}_1(\Delta)$ and $u(x) = x^2$, the following relation holds:

$$\inf_{v_h \in V_h} \int_0^1 |u'(x) - v_h'(x)|^2 dx = \frac{1}{3} h^2, \tag{3.28}$$

where $V_h = \text{span}\{p^{(i)} : i = 0, 1, ..., n\}$ is defined by linear finite elements on the mesh $0 = x^{(0)} < ... < x^{(i)} = ih < ... < x^{(n)} = 1$, $h = 1/n$.

34 Under the assumptions 1 and 2 of the Approximation Theorem 3.6, prove the completeness of the FE-spaces $\{V_h\}_{h \in \Theta}$ in $V = H^1(\Omega)$, i.e.,

$$\lim_{h \to 0} \inf_{v_h \in V_h} ||u - v_h|| = 0 \quad \forall u \in V. \tag{3.29}$$

35 Estimate (determine) the constant $c_A(\Delta)$ in the inequality

$$\max_{\xi \in \overline{\Delta}} |\sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)| \leq c_A(\Delta) \; ||\sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)||_{L_2(\Delta)} \tag{3.30}$$

from the proof of Lemma 3.11 for the linear triangular element. ($d = 2, k = 1, \mathcal{F}(\Delta) = \mathcal{P}_1$).

36* Let us assume that the weak solution u of the boundary value problem

$$-u''(x) = f(x), \; x \in (0,1) \quad \text{and} \quad u(0) = u(1) = 0$$

belongs to $W^2_\infty(0,1)$, and let u_h be its finite element approximation in the linear finite element space $V_{0,h} = \{v_h \in V_h : v_h(0) = v_h(1) = 0\}$, where V_h is defined above in Exercise 33, i.e.

$$u_h \in V_{0,h} : \int_0^1 u_h'(x) v_h'(x) dx = \int_0^1 f(x) v_h(x) dx, \; \forall v_h \in V_{0,h}.$$

Show that

$$\|u - u_h\|_{L_\infty(0,1)} = \|u - u_h\|_{C[0,1]} = O(h^2).$$