T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

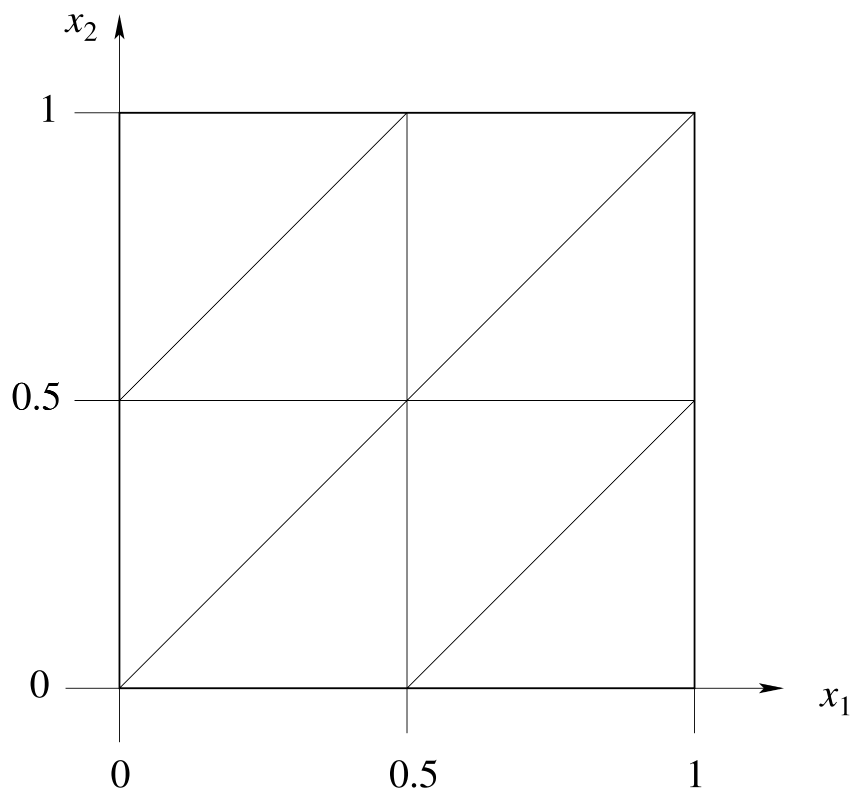
"Numerics of Elliptic Problems"

Tutorial 08 Tuesday, 20 May 2014, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 120.

28 Generate (by hand) the system of finite equations for the mixed boundary value problem

$$\begin{aligned}
-\Delta u(x_1, x_2) &= 0, \quad \forall\,(x_1, x_2) \in \Omega := (0,1) \times (0,1), && (3.17)\\
u(x_1, 0) &= 1 - x_1, \quad \forall\, x_1 \in [0,1], && (3.18)\\
u(x_1, 1) &= 0, \quad \forall\, x_1 \in [0,1], && (3.19)\\
u(0, x_2) &= 1 - x_2, \quad \forall\, x_2 \in [0,1], && (3.20)\\
u_{x_1}(1, x_2) &= x_2 - 1, \quad \forall\, x_2 \in (0,1). && (3.21)
\end{aligned}$$

assuming the triangulation shown in the figure below, where u_{x_1} denotes the partial derivatives of u with respect to x_1. Solve (by hand) the system of f.e. equations !



3.3 Properties of Systems of Finite Elements Equations

$\boxed{29}$ Prove the inheritance identity

$$(K_h \underline{u}_h, \underline{v}_h) = a(u_h, v_h) \quad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h} \ ! \tag{3.22}$$

$\boxed{30^\star}$ Show that the eigenvalue estimates from Theorem 2.4 are sharp with respect to h-order by proving the statement: there exist h-independent positive constants $\underline{c}'_E$ and $\overline{c}'_E$ satisfying the estimates:

$$\lambda_{\text{min}}\ (K_h) \le \underline{c}'_E h^d \quad \text{and} \quad \lambda_{\text{max}}(K_h) \ge \overline{c}'_E h^{d-2}. \tag{3.23}$$

Hint: Consider the special 1d ($d = 1$) boundary value problem $-u'' = f$ on $(0,1)$ with the Dirichlet boundary conditions $u(0) = u(1) = 0$, and use the Rayleigh quotient representation of the minimal and maximal eigenvalues !

$\boxed{31}$ Show that, for a regular triagulation according to Definition 2.3, there exist h-independent positive constants $\underline{c}_0$ and $\overline{c}_0$ satisfying the inequalities

$$\underline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \le (M_h \underline{v}_h, \underline{v}_h) \le \overline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \tag{3.24}$$

for all $\underline{v}_h \in \mathbf{R}^{N_h}$, where M_h denotes the mass matrix defined by the identity

$$(M_h \underline{u}_h, \underline{v}_h) := \int_\Omega u_h(x) v_h(x) dx \quad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h}. \tag{3.25}$$

$\boxed{32}$ Let $\lambda = \lambda_{\max}$ be the maximal eigenvalue of the generalized eigenvalue problem

$$K_h \underline{u}_h = \lambda M_h \underline{u}_h \tag{3.26}$$

and let $\lambda_r = \lambda_{r,\max}$ be the maximal eigenvalues of generalized eigenvalue problems

$$K_h^{(r)} \underline{u}_h^{(r)} = \lambda_r M_h^{(r)} \underline{u}_h^{(r)}, \tag{3.27}$$

where $K_h^{(r)}$ and $M_h^{(r)}$ denote the (local) element stiffness and mass matrices ($r = 1, 2, ..., R_h$), i.e., it holds

$$K_h = \sum_{r=1}^{R_h} C_r K_h^{(r)} C_r^T \quad \text{und} \quad M_h = \sum_{r=1}^{R_h} C_r M_h^{(r)} C_r^T.$$

Show the eigenvalue estimate

$$\lambda \le \max_{r=1,2,\ldots,R_h} \lambda_r. \tag{3.28}$$