T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 04 Tuesday, 08 April 2014, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 120.

17 Show that

$$\|u\|^*_{W_2^2(\Omega)} = \left(\int_{\Gamma_D} |u|^2 ds + \int_{\Gamma_D} |\partial_n u|^2 ds + |u|^2_{W_2^2(\Omega)}\right)^{1/2}$$

defines a new norm in $W_2^2(\Omega)$ that is equivalent to the standard norm

$$\|u\|_{W_2^2(\Omega)} = \left(\sum_{|\alpha|\le 2} \int_\Omega |\partial^\alpha u|^2 dx\right)^{1/2} = \left(\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + |u|^2_{W_2^2(\Omega)}\right)^{1/2},$$

where $\Gamma_D \subset \Gamma = \partial\Omega$ with $\text{meas}_{d-1}(\Gamma_D) > 0$, $\partial_n u(x) = \frac{\partial u}{\partial n}(x) = (\nabla u(x), n(x)) = \nabla u(x)^T n(x) = \nabla u(x) \bullet n(x)$, and $|u|_{W_2^2(\Omega)} = (\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^2 dx)^{1/2}$ denotes the standard semi-norm in $W_2^2(\Omega)$.

18 Show that there exists a positive constant $c_0 = const > 0$ and $c_1 = const > 0$ such that

$$\int_\Pi (u(x))^2 dx \le c_0 \left(\int_\Pi u(x)\, dx\right)^2 + c_1 \int_\Pi |\nabla u(x)|^2 dx \quad \forall u \in H^1(\Pi)$$

with $c_0 =?$ and $c_1 =?$, where $\Pi := \{x = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i < b_i,\ i = 1, 2\}$.

○ Hint: Use the representation

$$u(y_1, y_2) - u(x_1, x_2) = \int_{x_2}^{y_2} \frac{\partial u}{\partial \xi_2}(y_1, \xi_2) d\xi_2 + \int_{x_1}^{y_1} \frac{\partial u}{\partial \xi_1}(\xi_1, x_2) d\xi_1$$

19 Show that the inequalities

$$\inf_{q\in\mathbf{R}} \int_\Omega |u(x) - q|^2 dx \le c^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

and

$$\int_\Omega |u(x)|^2 dx \le \frac{1}{|\Omega|} \left(\int_\Omega u(x)\, dx\right)^2 + c^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

are equivalent !

$\boxed{20}$ Let us consider the quadrature rule

$$\int_\Delta u(\xi)d\xi \approx u(\xi^*)|\Delta|,$$

with the unit triangle $\Delta = \{\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 : 0 < \xi_2 < 1 - \xi_1, 0 < \xi_1 < 1\}$ and the integration point $\xi^* = (1/3, 1/3)$. Show that there exists a positive constant $c = const. > 0$ such that

$$|\int_\Delta u(\xi)d\xi - u(\xi^*)|\Delta|\,| \leq c\,|u|_{H^2(\Delta)} \quad \forall u \in H^2(\Delta).$$

Hint: In 2D ($d = 2$), $H^2(\Delta)$ is continuously (even compactly) embedded in $C(\overline{\Delta})$, i.e. there exists $c_E = const. > 0 :\ \|u\|_{C(\overline{\Delta})} := \max_{\xi\in\Delta} |u(\xi)| \leq c_E \|u\|_{H^2(\Delta)}$.

$\boxed{21}$ Show that, for sufficiently smooth functions, e.g. for $u, v \in H(curl) \cap [C^1(\overline{\Omega})]^3$, the curl-IbyP-formula

$$\int_\Omega \mathrm{curl}(u) \cdot v\, dx = \int_\Omega u \cdot \mathrm{curl}(v)\, dx - \int_\Gamma (u \times n) \cdot v\, ds \tag{2.12}$$

is valid. **Hint:** Use the classical IbyP-formula for the proof of (2.12) !