T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 03 Tuesday, 1 April 2014, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 120.

1.3 Scalar elliptic problems of the fourth order

11 Show that the first biharmonic BVP

$$u \in V_0 := H_0^2(\Omega) : \int_\Omega \Delta u(x) \Delta v(x) dx = \int_\Omega f(x) v(x) dx \; \forall v \in V_0 \tag{1.9}$$

allows the application of Lax-Milgram-Theorem. Then formulate a minimization problem that is equivalent to the variational formulation above.

12* Give the variational formulations for BVPs of the second, the third and the fourth kind mentioned in Remark 1.6.2 and discuss the existence and uniqueness of generalized solutions. Without loss of generality, consider homogenized essential BCs only.

13* For the Kirchhoff-Plate, the plate bilinear form

$$a(u,v) := \int_\Omega \left\{ \Delta u(x) \Delta v(x) + (1-\sigma) \left[2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right] \right\} dx \tag{1.10}$$

is only identical to the biharmonic bilinear form given in (1.9) in the case of the first BVP (i.e., on $H_0^2(\Omega)$). Prove this under the assumption that $\sigma \in (0,1)$ is a given material parameter (Poisson-coefficient).

14* Which natural BCs should be impossed for the plate bilinear form (1.10) ?

2 Tools from the Theory of Sobolev Spaces

15 Let us consider the function

$$u(x) = \begin{cases} 1, & -1 \le x \le 0 \\ -1, & 0 \le x \le -1 \end{cases},$$

Obviously, $u \in L_p(\Omega) \subset L_{loc}(\Omega) \subset D'(\Omega)$, but $u \notin C(\overline{\Omega})$!
Compute

1. $u' = \partial^1 u \in$?
2. $u'' = \partial^2 u \in$?
3. $u''' = \partial^3 u \in$?

in the distributive sense !

$\boxed{16}$ Show that

$$|g|_{H^{1/2}(\Gamma)} = \inf_{u \in H^1(\Omega):\, \gamma_0 u = g} |u|_{H^1(\Omega)} \tag{2.11}$$

defines a semi-norm in $H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega)$ (check the semi-norm axioms), where $|u|_{H^1(\Omega)} := \|\nabla u\|_{L_2(\Omega)}$ denotes the standard semi-norm in $H^1(\Omega)$! The infimum in (2.11) is realized. Characterize the minimizer $u^* \in H^1(\Omega)$ as a unique solution of a variational problem !