T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 02 Tuesday, 25 March 2014, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: S2 120.

1.2 The linear elasticity problem

$\boxed{07}$ Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) and for the mixed BVP ($\text{meas}_2(\Gamma_1) > 0$ and $\text{meas}_2(\Gamma_2) > 0$) of the linear elasticity. the following statements are true:

1. $a(.,.)$ is symmetric, i.e., $a(u,v) = a(v,u) \quad \forall u, v \in V$,
2. $a(.,.)$ is nonnegative, i.e., $a(v,v) \geq 0 \quad \forall v \in V$,
3. $a(.,.)$ is possitive on $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$, if $\text{meas}_2(\Gamma_1) > 0$, i.e., $a(v,v) > 0 \quad \forall v \in V_0 : v \not\equiv 0$.

The equivalence of VF $(9)_{VF}$ and MP $(9)_{MP}$ then follows from the statements 1. and 2. above according to Section. 1.1. of the lecture.

$\boxed{08}$ Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) of 3D linear elasticity in the case of isotrop and homogeneous material, the assumptions of Lax-Milgram-Theorem are fulfiled. Provide constants μ_1 and μ_2 such that:

1) $F \in V_0^*$,

2a) $\exists\, \mu_1 = \text{const} > 0 : \; a(v,v) \geq \mu_1 \parallel v \parallel^2_{H^1(\Omega)} \; \forall v \in V_0$,

2b) $\exists\, \mu_2 = \text{const} > 0 : \; |a(u,v)| \leq \mu_2 \parallel u \parallel^2_{H^1(\Omega)} \parallel v \parallel^2_{H^1(\Omega)} \; \forall u, v \in V_0$.

○ Hint: to the proof of V_0-ellipticity:

- $a(v,v) \geq 2\mu \int_\Omega \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx$,
- Korn's inequality for the BVP of the first type: $V_0 = [H_0^1(\Omega)]^3$, where $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ auf } \Gamma\}$,
- FRIEDRICHS-inequality.

$\boxed{09}$ Formulate the iterative method (3) from Section 1.1 of the lecture for the first BVP of the linear elasticity in case of 3D homogeneous and isotrop material, i.e.,

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = (H_0^1(\Omega))^3, \tag{1.6}$$

for $n = 0, 1, 2, ...$, and given $u_0 \in V_0$. Derive the weak form, i.e., the variational formulation, for the calculation of $u_{n+1} \in V_0$. Discuss two cases in which the scalar product in V_0 is defined as follows:

$$(u, v)^2_{V_0} := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V_0, \tag{1.7}$$

and

$$(u, v)^2_{V_0} := \int_\Omega (\nabla u \cdot \nabla v + uv) \, dx \quad \forall u, v \in V_0. \tag{1.8}$$

$\boxed{10^*}$ Let us consider the variational formulation:
(*) Find $u \in V_g = V_0 : \quad a(u, v) = \langle F, v \rangle \quad \forall v \in V_0$ of a plane linear elasticity problem in $\Omega = (0, 1) \times (0, 1)$, where

$$\begin{aligned}
V_0 = & \; \{ \; u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\
& \quad u_1 = 0 \text{ on } \Gamma_2 = \{0\} \times [0, 1] \cup \{1\} \times [0, 1], \\
& \quad u_2 = 0 \text{ on } \Gamma_1 = [0, 1] \times \{0\} \cup [0, 1] \times \{1\} \}, \\
a(u, v) \; &= \int_\Omega D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx = \int_\Omega \sigma_{kl}(u) \varepsilon_{kl}(v) \, dx, \\
\langle F, v \rangle \; &= \int_\Omega f_i v_i \, dx + \int_{\Gamma_1} ? \, ds + \int_{\Gamma_2} ? \, ds.
\end{aligned}$$

Impose the right natural BCs! Give the classical formulation of (*).