

# T U T O R I A L

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 01**

Thursday, 18 March 2014, Time: 10<sup>15</sup> – 11<sup>45</sup>, Room: S2 120.

## 1 Variational formulation of multi-dimensional elliptic Boundary Value Problems (BVP)

### 1.1 Scalar Second-order Elliptic BVP

○ In Section 1.2.1 of our lectures, we considered the BVP in classical formulation

$$\begin{aligned}
 &\text{Find } u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
 & - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = f(x), x \in \Omega \\
 & +\text{BC: } \bullet \quad u(x) = g_1(x), x \in \Gamma_1 \\
 & \bullet \quad \frac{\partial u}{\partial N} := \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u(x)}{\partial x_j} n_i(x) = g_2(x), x \in \Gamma_2 \\
 & \bullet \quad \frac{\partial u}{\partial N} + \alpha(x)u(x) = \underbrace{g_3(x)}_{\alpha(x)u_A(x)}, x \in \Gamma_3
 \end{aligned} \tag{1.1}$$

and derived the variational formulation

$$\begin{aligned}
 &\text{Find } u \in V_g \text{ such that } a(u, v) = \langle F, v \rangle \quad \forall v \in V_0, \\
 & \text{with} \\
 & a(u, v) := \int_{\Omega} \left( \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx + \int_{\Gamma_3} \alpha uv ds, \\
 & \langle F, v \rangle := \int_{\Omega} f v dx + \int_{\Gamma_2} g_2 v ds + \int_{\Gamma_3} g_3 v ds, \\
 & V_g := \{v \in V = W_2^1(\Omega) : v = g_1 \text{ on } \Gamma_1\}, \\
 & V_0 := \{v \in V : v = 0 \text{ on } \Gamma_1\}.
 \end{aligned} \tag{1.2}$$

under the assumptions

$$\begin{array}{l}
 1) \quad a_{ij}, b_i, c \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3) \\
 2) \quad f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2, 3 \\
 3) \quad g_1 \in H^{\frac{1}{2}}(\Gamma_1), \text{ i.e., } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1. \\
 4) \quad \Omega \subset \mathbf{R}^d (\text{bounded}) : \Gamma = \partial\Omega \in C^{0,1} \text{ (Lip boundary)} \\
 5) \quad \text{uniform ellipticity:} \\
 \left. \begin{array}{l}
 \sum_{i,j=1}^d a_{ij}(x)\xi_i; \xi_j \geq \bar{\mu}_1 |\xi|^2 \quad \forall \xi \in \mathbf{R}^d \\
 a_{ij}(x) = a_{ji}(x) \quad \forall i, j = \overline{1, d}
 \end{array} \right\} \forall \text{ a.e. } x \in \Omega.
 \end{array} \tag{1.3}$$

**01** Formulate the classical assumptions on  $\{a_{ij}, b_i, c, \alpha, f, g_i, \Omega \text{ resp. } \partial\Omega\}$  for (1.1) !

**02** Show that, for sufficiently smooth data, a the generalized solution  $u \in V_g \cap X \cap H^2(\Omega)$  of the Boundary Value Problem (2) is also a classical solution, i.e. a solution of (1) !

$$(1) \quad \left\{ \begin{array}{l}
 \text{Find } u \in X = C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
 -\Delta u(x) + u(x) = f(x), x \in \Omega \subset \mathbf{R}^d \text{ (bounded)}, \\
 u(x) = g_1(x), x \in \Gamma_1, \\
 \frac{\partial u}{\partial n}(x) = g_2(x), x \in \Gamma_2, \\
 \frac{\partial u}{\partial n}(x) = \alpha(x)(g_3(x) - u(x)), x \in \Gamma_3
 \end{array} \right.$$

? $\Downarrow$  $\Uparrow$ ?

$$(2) \quad \left\{ \begin{array}{l}
 \text{Find } u \in V_g = \{v \in V = H^1(\Omega) : v = g_1 \text{ on } \Gamma_1\} \text{ such that } \forall v \in V_0 : \\
 \underbrace{\int_{\Omega} (\nabla^T u \nabla v + uv) dx + \int_{\Gamma_3} \alpha uv ds}_{=a(u,v)} = \underbrace{\int_{\Omega} f v dx + \int_{\Gamma_2} g_2 v ds + \int_{\Gamma_3} \alpha g_3 v ds}_{= \langle F, v \rangle}
 \end{array} \right.$$

where  $V_0 = \{v \in V = H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$ .

**03** Show that the assumptions of the Lax-Milgram-Theorem are satisfied for the variational problem (1.2) under the assumptions (1.3) and the additional assumptions  $b_i = 0, c = 0, \alpha(x) \geq \underline{\alpha} = \text{const} > 0 \quad \forall \text{ a.e. } x \in \Gamma_3$ , and  $\text{meas}_{d-1}(\Gamma_i) > 0, i = 1, 2, 3$  !

**04** In addition to assumption (1.3), let us assume that  $c(x) \geq \underline{c} = \text{const} > 0$  for almost all  $x \in \Omega, \Gamma_1 = \Gamma_3 = \emptyset$ , and  $b_i \neq 0$ . Provide conditions for the coefficients  $b_i(\cdot)$  such that the assumptions of the Lax-Milgram-Theorem are satisfied !

○ Hint: For the estimate of the convection term  $\sum_{i=1}^d \int_{\Omega} b_i \frac{\partial u}{\partial x_i} v dx$ , make use of the  $\varepsilon$ -inequality (Young's inequality)

$$|ab| \leq \frac{1}{2\varepsilon} a^2 + \varepsilon b^2, \quad \forall a, b \in \mathbf{R}^1 \quad \forall \varepsilon > 0 !$$

**05** Derive the variational formulation of the pure Neumann problem for the Poisson equation

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma := \partial\Omega, \quad (1.4)$$

and discuss the question of the existence and uniqueness of a generalized solution of (1.4) !

○ Hint:

Obviously,  $u(x) + c$  with an arbitrary constant  $c \in \mathbb{R}^1$  solves (1.4) provided that  $u$  is the solution of the BVP (1.4). There are the following ways to analyze the existence of a generalized solution:

- 1) Set up the variational formulation in  $V = H^1(\Omega)$  and apply the FREDHOLM-Theory !
- 2) Set up the variational formulation in the factor-space  $V = H^1(\Omega)|_{\ker}$  with  $\ker = \{c : c \in \mathbb{R}^1\} = \mathbb{R}^1$  and apply the LAX-MILGRAM-Theorem !

**06\*** Derive the variational formulation of the Dirichlet problem for the Helmholtz equation

$$-\Delta u - \omega^2 u = f \text{ in } \Omega = (0, 1)^2 \subset \mathbb{R}^2 \quad \text{and} \quad u = 0 \text{ on } \Gamma := \partial\Omega, \quad (1.5)$$

where  $\omega^2$  is a given positive constant. Then discuss the problem of the existence and uniqueness of a generalized solution of (1.5) !