Zenger Correction

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Outline

1. Decomposition of the error
   - Estimation of the error terms
2. Theorem: Existence Zenger Correction s.t. $u(x) - R_{h,\gamma}u(x) = O(h^2|\log(h)|)$

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Decomposition of the error

$$u(x) - R_{h,\gamma} u(x) = a(u - R_{h,\gamma} u, g_x) + O(h^2)$$

Expansion of u

$$u(x) = c\,\phi(x) + U(x), \quad U(x) \in H^2(\Omega)$$

$$\phi(x) - R_{h,\gamma}\phi(x) = a(\phi - R_{h,\gamma}\phi, g_x) + O(h^2)$$

Decompose regularized Green's function g_x and $R_{h,\gamma} g_x$

$$g_x(y) = d(x)\,\phi(y) + G_x(y)$$
$$R_{h,\gamma} g_x(y) = d(x)\,R_{h,\gamma}\phi(y) + R_{h,\gamma} G_x(y)$$

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Modified Orthogonality

$$a(\phi - R_{h,\gamma}\phi, v_h) + K_\gamma(R_{h,\gamma}\phi, v_h) = 0, \quad \forall v_h \in V_{h,0}$$

$$\begin{aligned}
=& a(\phi - R_{h,\gamma}\phi, g_x) + a(\phi - R_{h,\gamma}\phi, -R_{h,\gamma} g_x) \\
& \qquad + K_\gamma(R_{h,\gamma}\phi, -R_{h,\gamma} g_x) + O(h^2) \\
=& a(\phi - R_{h,\gamma}\phi, g_x - R_{h,\gamma} g_x) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma} g_x) + O(h^2)
\end{aligned}$$

$$\begin{aligned}
=& a(\phi - R_{h,\gamma}\phi, \left[d(x)\,\phi + G_x\right] - \left[d(x)\,R_{h,\gamma}\phi + R_{h,\gamma} G_x\right]) \\
& - K_\gamma(R_{h,\gamma}\phi, \left[d(x)\,R_{h,\gamma}\phi + R_{h,\gamma} G_x\right]) + O(h^2)
\end{aligned}$$

$$\begin{aligned}
=& d(x)\left[a(\phi - R_{h,\gamma}\phi, \phi - R_{h,\gamma}\phi) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi)\right] \\
& + a(\phi - R_{h,\gamma}\phi, G_x - R_{h,\gamma} G_x) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma} G_x) + O(h^2)
\end{aligned}$$

Estimation first error term

$$a(\phi - R_{h,\gamma}\phi, \phi - R_{h,\gamma}\phi) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi)$$

Lemma

If there exists a ξ such that

$$a(\phi - P_h\phi, \phi - P_h\phi) - K_\xi(P_h\phi, P_h\phi) = 0$$

and $a_\eta(u, v) = a(u, v) - K_\eta(u, v)$ is positive definite $\forall\, 0 \leq \eta \leq \xi$

then there is a γ with $0 \leq \gamma \leq \xi$, such that

$$\begin{aligned} & a(\phi, \phi) - a_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi) \\ & = a(\phi - R_{h,\gamma}\phi, \phi - R_{h,\gamma}\phi) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi) = O(h^2) \end{aligned}$$

Modified Energy depends continuously on γ:

$$a(\phi - R_{h,\gamma}\phi, \phi - R_{h,\gamma}\phi) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi)$$

is a continuous function of γ.

Minimization property

$R_{h,\xi}\phi$ is the minimizer of

$$\min_{\phi_h \in V_{h,0}} a(\phi - \phi_h, \phi - \phi_h) - K_\xi(\phi_h, \phi_h)$$

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$\Rightarrow$ **Intermediate value theorem**

Assumption: There exists a ξ s.t.

$$a(\phi - P_h\phi, \phi - P_h\phi) - K_\xi(P_h\phi, P_h\phi) = 0$$

- Behaviour of ϕ only depends on inner angle of the reentrant corner.
  Behaviour of P_h only depends on the triangulation.
- For a local correction

$$K_\xi(u_h, v_h) = \xi \sum_{i,j\in N(0,0)} \beta_{ij} u_h(x_i) v_h(x_j)$$

  we receive a linear equation for $\xi \to$ unique solution.
  Problem: ξ can be too large, s.t. $a_\xi(u, v) = a(u, v) - K_\xi(u, v)$ is not positive definite.
- E.g. regular triangulation (triangles are of the same size) the corresponding $a_\xi(u, v)$ is always positive definite.

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Estimation second error term

$$a(\phi - R_{h,\gamma}\phi, G_x - R_{h,\gamma}G_x) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}G_x)$$

Lemma

For g_x decomposed in the following way $g_x(y) = d(x)\,\phi(y) + G_x(y)$ we have

$$a(\phi - R_{h,\gamma}\phi, G_x - R_{h,\gamma}G_x) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}G_x) = O(h^2|\log(h)|).$$

[Proof: Blum and Rannacher, Extrapolation Techniques for Reducing the Pollution Effect of Reentrant Corners in the Finite Element Method.]

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2. Theorem: Existence Zenger Correction s.t. $u(x) - R_{h,\gamma}u(x) = O(h^2|\log(h)|)$

Theorem

Let Ω be an L-shaped domain and the triangulation be regular and non-degenerated.
Find $u \in H_0^1(\Omega)$ s.t. $a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega)$.
Let

$$K(u_h, v_h) = \sum_{i,j \in N(0,0)} \beta_{ij} u_h(x_i) v_h(x_j)$$

be a local correction which is positive semidefinite and ξ be in $\mathbb{R}_+$ s.t.

$$a_\xi(u,v) = a(u,v) - \xi K(u,v)$$

is positive definite and

$$a(\phi - P_h\phi, \phi - P_h\phi) - K_\xi(P_h\phi, P_h\phi) = 0$$

Theorem (cont.)

Then there exists a correction parameter $0 \leq \gamma \leq \xi$
s.t. the numerical solution $R_{h,\gamma}u$ *of the corrected problem fulfils the pointwise estimate*

$$u(x) - R_{h,\gamma}u(x) = O(h^2|\log(h)|) \quad \textit{for fixed } x \in \Omega.$$

Decompose u and $R_{h,\gamma}u$ into a singular and regular component.

$$u(x) = c\,\phi(x) + U(x), \quad U \in H^2(\Omega)$$
$$R_{h,\gamma}u(x) = c\,R_{h,\gamma}\phi(x) + R_{h,\gamma}U(x)$$

Pointwise error in the singular component

$$\begin{aligned}&\phi(x) - R_{h,\gamma}\phi(x)\\&= d(x)\big[a(\phi - R_{h,\gamma}\phi, \phi - R_{h,\gamma}\phi) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}\phi)\big]\\&+ a(\phi - R_{h,\gamma}\phi, G_x - R_{h,\gamma}G_x) - K_\gamma(R_{h,\gamma}\phi, R_{h,\gamma}G_x) + O(h^2)\end{aligned}$$

- 1^{st} error term is of order $O(h^2)$ (γ chosen accordingly).
- 2^{nd} error term is of order $O(h^2|\log(h)|)$ independently of γ.

Pointwise error in the regular component

$$U(x) - R_{h,\gamma}U(x) = O(h^2|\log(h)|).$$

[Proof: Blum and Rannacher, Extrapolation Techniques for Reducing the Pollution Effect of Reentrant Corners in the Finite Element Method.]

Remark: Generalization of theorem

The above theorem is also valid for the more general case where decomposition of u consists of more then one singular function.

Concrete example of a Zenger Correction

Special case: L-shaped domain + regular 5-point-discretization

Possible correction

Replace the difference equation

$$u_h(0,0) = g(0,0) \quad \text{by}$$

$$u_h(0,0) - g(0,0) = \frac{1}{\kappa}\left[u_h(-h,0) - 2g(0,0) + g(h,0)\right]$$

where $\kappa = 5.07925\ldots$.

$$u(x) - u_h(x) = O(h^2) \quad \text{for fixed } x \in \Omega$$

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