Variational Solutions and Classical Regularity Results

Katharina Rafetseder

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Outline

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1. Variational solution

2. Local regularity
   - Interior regularity
   - Local regularity up to the boundary away from vertices and in 3d in addition edges

Classical formulation

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_j,\ j \in \mathcal{D} \\ \frac{\partial u}{\partial \nu_j} &= 0 \quad \text{on } \Gamma_j,\ j \in \mathcal{N} \end{aligned}$$

Variational formulation
Find $u \in V := \{u \in H^1(\Omega) : \gamma_j u = 0 \text{ on } \Gamma_j,\ j \in \mathcal{D}\}$

$$\int_\Omega \nabla u . \nabla v \, dx = -\int_\Omega f v \, dx, \quad \forall v \in V$$

- $\mathcal{D}$ not empty

$$c_1|v|_{H^1(\Omega)} \leq \|v\|_{H^1(\Omega)} \leq c_2|v|_{H^1(\Omega)}, \quad \forall v \in V$$

V is a Hilbert space for the scalar product

$$\int_\Omega \nabla u . \nabla v \, dx = a(u, v)$$

Riesz Theorem

$$l(v) = -\int_\Omega fv \, dx$$

Bounded linear functional on V (assume $f \in L^2(\Omega)$)

$$!\exists u \in V : a(u, v) = -\int_\Omega fv \, dx, \quad \forall v \in V$$

- $\mathcal{D}$ empty
  Replace V by factor space $V/\mathbb{R}$

In what sense u fulfills the classical formulation?

1.
$$
\begin{aligned}
-\int_\Omega \nabla u . \nabla v \, dx &= \int_\Omega f v \, dx, \quad \forall v \in \mathcal{D}(\Omega) = C_C^\infty(\Omega) \\
< \Delta u, v > &= < f, v >, \quad \forall v \in \mathcal{D}(\Omega) \\
\Delta u &= f \quad \text{in } \mathcal{D}'(\Omega)
\end{aligned}
$$

   Since $u \in H^1(\Omega)$ and its distributive Laplace $\Delta u = f$ is a regular distribution

$$\Delta u = f \quad \text{in } L^2(\Omega)$$

2. Dirichlet boundary conditions: $\gamma_j u = 0$ on $\Gamma_j, j \in \mathcal{D}$
3. Neumann boundary conditions: not obvious so far

Neumann boundary conditions in 2d

$\Delta u = f \quad \text{in } L^2(\Omega)$
$u \in E(\Delta, L^2(\Omega)) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$

$$\int_\Omega \nabla u . \nabla v \, dx = -\int_\Omega vf \, dx + \sum_j < \gamma_j \left(\frac{\partial u}{\partial \nu_j} \right), \gamma_j v >_{\tilde{H}^{-1/2}(\Gamma_j) x \tilde{H}^{1/2}(\Gamma_j)}$$

$$\forall v \in V \text{ s.t. } \gamma_j v \in \tilde{H}^{1/2}(\Gamma_j), \forall j$$

$$\sum_j < \gamma_j \left(\frac{\partial u}{\partial \nu_j} \right), \gamma_j v >_{\tilde{H}^{-1/2}(\Gamma_j) x \tilde{H}^{1/2}(\Gamma_j)} = 0$$

$j \in \mathcal{N}$: $\quad \gamma_j v$ is an arbitrary function in $\tilde{H}^{1/2}(\Gamma_j)$

$$\gamma_j \left(\frac{\partial u}{\partial \nu_j} \right) = 0 \quad \text{in } \tilde{H}^{-1/2}(\Gamma_j), \forall j \in \mathcal{N}$$

Theorem

Ω polygonal domain in $\mathbb{R}^2$, $u \in H^1(\Omega)$ fulfills conditions:

- $\Delta u = f \quad$ *in* $L^2(\Omega)$
- $\gamma_j u = 0 \quad$ *on* Γ_j, $j \in \mathcal{D}$
- $\gamma_j(\frac{\partial u}{\partial \nu_j}) = 0 \quad$ *in* $\tilde{H}^{-1/2}(\Gamma_j)$, $j \in \mathcal{N}$

$u \in V$ and is unique variational solution of

$$\int_\Omega \nabla u . \nabla v \, dx = - \int_\Omega f v \, dx, \quad \forall v \in V$$

Lemma

The subspace of functions in V which vanish near the corners is dense in V.

For $v \in V$, which vanishes near the corners

$$\int_\Omega \nabla u . \nabla v \, dx = - \int_\Omega \Delta u v \, dx + \sum_j < \gamma_j(\frac{\partial u}{\partial \nu_j}), \gamma_j v >_{\tilde{H}^{-1/2}(\Gamma_j) x \tilde{H}^{1/2}(\Gamma_j)}$$

$$= - \int_\Omega f v \, dx + \sum_j \begin{cases} 0, & j \in \mathcal{D} \quad (\gamma_j v = 0) \\ 0, & j \in \mathcal{N} \quad (\gamma_j(\frac{\partial u}{\partial \nu_j}) = 0 \text{ in } \tilde{H}^{-1/2}(\Gamma_j)) \end{cases}$$

Proof of Lemma

Know $H^2(\Omega) \cap V$ is dense in V.
$\Rightarrow$ Approximate function $v \in H^2(\Omega) \cap V$ by functions which in addition vanish near the corners

Truncation near the corners:

- Truncation function

$$\psi_m(X) := \prod_j \psi(m \underbrace{r_j(X)}_{:=|X-S_j|}),\ \psi \in C^\infty([0,\infty[) + \text{truncation prop.}$$

- ψ_m vanishes near the corners.
- $\psi_m(X) = 1$ if $r_j(X) > \frac{1}{m}, \forall m$
- $|\psi_m| \leq 1, \quad |\nabla \psi_m| \leq Km, \quad supp(\nabla \psi_m) \subseteq \bigcup_j \mathcal{B}(S_j, \frac{1}{m})$

- $v \in H^2(\Omega) \cap V$ define $v_m := \psi_m v$
  $\to$ functions v_m vanish near corners
  - $\forall m : \|v_m\|_{H^1(\Omega)} \leq C$ (bounded in V)
- V Hilbert space: Every bounded sequence in V contains a weakly convergent subsequence.

$$\forall v \in H^2(\Omega) \cap V \quad \exists (v_{m_k}) : v_{m_k} \rightharpoonup v$$

- Sobolev's Embedding Theorem

$$v \in H^2(\Omega) \hookrightarrow C(\bar{\Omega})$$

weak density $\Longleftrightarrow$ strong density

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1 Variational solution

2 Local regularity

- Interior regularity
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Lemma

Let $u \in H^1(\mathbb{R}^n)$ and $\Delta u \in L^2(\mathbb{R}^n)$ then $u \in H^2(\mathbb{R}^n)$.

Plancherel Theorem

$$\|\mathcal{F}g\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)}, \quad \forall g \in L^2(\mathbb{R}^n)$$
$$\|\mathcal{F}\Delta u\|_{L^2(\mathbb{R}^n)} = \|\Delta u\|_{L^2(\mathbb{R}^n)} < \infty$$

Fourier transform of a differential operator with const. coefficients

$$\mathcal{F}(L(D)u)(\rho) = L(\mathrm{i}\rho)\mathcal{F}u(\rho), \quad L \text{ symbol of the differential operator}$$

$$\mathcal{F}(\Delta u)(\rho) = -\sum_{i=1}^{n} {\rho_i}^2 \mathcal{F}u(\rho) = -|\rho|^2 \mathcal{F}u(\rho)$$

$$\|-|\rho|^2\mathcal{F}u(\rho)\|^2{}_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} {\rho_i}^2\right)^2 |\mathcal{F}u(\rho)|^2 \, d\rho = \|\Delta u\|^2{}_{L^2(\mathbb{R}^n)} < \infty$$

H^2 semi-norm

$$\|\nabla^2 u\|^2{}_{L^2(\mathbb{R}^n)} = \|\mathcal{F}\nabla^2 u\|^2{}_{L^2(\mathbb{R}^n)}$$
$$\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \left|\frac{\partial u}{\partial x_i \partial x_j}\right|^2 dx = \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \left|\mathcal{F}\frac{\partial u}{\partial x_i \partial x_j}\right|^2 d\rho$$

Fourier transform of derivatives

$$\mathcal{F}(D^\alpha u)(\rho) = \mathrm{i}^{|\alpha|}\rho^\alpha \mathcal{F}u(\rho)$$
$$\mathcal{F}\left(\frac{\partial u}{\partial x_i \partial x_j}\right)(\rho) = \mathrm{i}^2 \rho_i \rho_j \mathcal{F}u(\rho)$$

$$= \int_{\mathbb{R}^n} \left(\sum_{i,j=1}^{n} \rho_i{}^2 \rho_j{}^2\right) |\mathcal{F}u(\rho)|^2 \, d\rho = \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n} \rho_i{}^2\right)^2 |\mathcal{F}u(\rho)|^2 \, d\rho < \infty$$

Theorem (Interior regularity)

Let $u \in V$ be the variational solution of

$$\int_\Omega \nabla u . \nabla v \, dx = - \int_\Omega f v \, dx, \quad \forall v \in V$$

then $\varphi u \in H^2(\Omega), \forall \varphi \in \mathcal{D}(\Omega)$.

$$V := \{ v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j, j \in \mathcal{D} \}$$

$w := \varphi u, \; \varphi \in \mathcal{D}(\Omega)$

$\tilde{w} \in H^1(\mathbb{R}^n)$ ($\tilde{w}$ continuation of w by 0 outside Ω)

For $v \in H^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \nabla \tilde{w}.\nabla v \, dx = \int_\Omega \nabla(\varphi u).\nabla v \, dx = \underbrace{\int_\Omega u \nabla\varphi.\nabla v \, dx}_{I} + \underbrace{\int_\Omega \varphi \nabla u.\nabla v \, dx}_{II}$$

$$I = -\int_\Omega \nabla.(u\nabla\varphi) v \, dx + \underbrace{\int_\Gamma \gamma u \gamma v \nabla\varphi.\nu \, d\sigma}_{=0} = -\int_\Omega (\nabla u.\nabla\varphi + u\Delta\varphi) v \, dx$$

u variat. sol., $\forall v \in V := \{v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j, j \in \mathcal{D}\}$

$$\int_\Omega \nabla u . \nabla v \, dx = - \int_\Omega f v \, dx$$

$v \in H^1(\mathbb{R}^n)$, $v' := \varphi v \in V$

$$\int_\Omega \nabla u . \nabla(\varphi v) \, dx = \int_\Omega v \nabla u . \nabla \varphi + \underbrace{\varphi \nabla u . \nabla v}_{II} \, dx = - \int_\Omega f \varphi v \, dx$$

$$\int_{\mathbb{R}^n} \nabla \tilde{w} . \nabla v \, dx = - \int_\Omega \underbrace{(2 \nabla u . \nabla \varphi + u \Delta \varphi + f \varphi)}_{=:g \in L^2(\Omega), \quad \tilde{g} \in L^2(\mathbb{R}^n)} v \, dx, \quad \forall v \in \underbrace{H^1(\mathbb{R}^n)}_{\supseteq \mathcal{D}(\mathbb{R}^n)}$$

$\Delta \tilde{w} = \tilde{g} \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \Rightarrow \Delta \tilde{w} = \tilde{g} \quad \text{in } L^2(\mathbb{R}^n)$

Lemma: $\tilde{w} \in H^2(\mathbb{R}^n) \Rightarrow w = \varphi u \in H^2(\Omega)$

Theorem (Local regularity up to the boundary away from vertices and in 3d in addition edges)

Let $u \in V$ be the variational solution of

$$\int_\Omega \nabla u . \nabla v \, dx = -\int_\Omega f v \, dx, \quad \forall v \in V$$

then $\varphi u \in H^2(\Omega), \forall \varphi \in \mathcal{D}(\bar{\Omega})$ whose support intersects Γ only at interior points of the Γ_j.

$\mathcal{D}(\bar{\Omega}) :=$ restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$.

Only proof a special case:

$$\frac{\partial \varphi}{\partial \nu} = \nabla \varphi . \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\frac{\partial \varphi}{\partial x_n} = 0 \quad \text{on } x_n = 0$$

Case: $k \in \mathcal{N}$

$w := \varphi u$

$\tilde{w} \in H^1(H := \{x_n > 0\})$ ($\tilde{w}$ continuation of w by 0 outside Ω)

For $v \in H^1(H)$:

$$\int_H \nabla \tilde{w} . \nabla v \, dx = \int_\Omega \nabla(\varphi u) . \nabla v \, dx = \underbrace{\int_\Omega u \nabla \varphi . \nabla v \, dx}_{I} + \underbrace{\int_\Omega \varphi \nabla u . \nabla v \, dx}_{II}$$

$$I = - \int_\Omega \nabla . (u \nabla \varphi) v \, dx + \underbrace{\sum_j \int_{\Gamma_j} \gamma_j u \gamma_j v \nabla \varphi . \nu_j \, d\sigma}_{= \int_{\Gamma_k} \gamma_k u \gamma_k v \nabla \varphi . \nu_k \, d\sigma = 0}$$

u variat. sol., $\forall v \in V := \{v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j, j \in \mathcal{D}\}$

$$\int_\Omega \nabla u . \nabla v \, dx = -\int_\Omega fv \, dx$$

$v \in H^1(H)$, $v' := \varphi v \in V$ $\quad$ ($supp(\varphi)$ only meets a Neumann boundary)

$$\int_\Omega \nabla u . \nabla(\varphi v) \, dx = \int_\Omega v \nabla u . \nabla \varphi + \underbrace{\varphi \nabla u . \nabla v}_{//} \, dx = -\int_\Omega f \varphi v \, dx$$

$$(*) \quad \int_H \nabla \tilde{w} . \nabla v \, dx = -\int_\Omega \underbrace{(2\nabla u . \nabla \varphi + u \Delta \varphi + f \varphi)}_{:= g \in L^2(\Omega), \quad \tilde{g} \in L^2(H)} v \, dx, \quad \forall v \in H^1(H)$$

Even reflection through $\{x_n = 0\}$

$$W(x', x_n) := \begin{cases} \tilde{w}(x', x_n) & , x_n > 0 \\ \tilde{w}(x', -x_n) & , x_n < 0 \end{cases}$$

$W \in H^1(\mathbb{R}^n)$?

$$\nabla W(x', x_n) = \begin{cases} \nabla \tilde{w}(x', x_n) & , x_n > 0 \\ \nabla(\tilde{w}(x', -x_n)) & , x_n < 0 \end{cases}$$

$$\psi \in \mathcal{D}(\mathbb{R}^n)^3 : \quad \int_{\mathbb{R}^n} \nabla W(x', x_n).\psi \, dx = \int_{\{x_n > 0\}} \dots + \int_{\{x_n < 0\}} \dots =$$

$$= - \int_{\mathbb{R}^n} W(x', x_n) \nabla.\psi \, dx + \int_{\{x_n = 0\}} (\gamma \tilde{w}(x', x_n) - \gamma \tilde{w}(x', -x_n))\psi.\nu \, d\sigma$$

$\gamma \tilde{w}(x', x_n) = \gamma \tilde{w}(x', -x_n)$ on $x_n = 0$ ✓

$$\nabla(\tilde{w}(x', -x_n)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (\nabla_z \tilde{w})(x', \underbrace{-x_n}_{z})$$

For $v \in H^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \nabla W . \nabla v \, dx = \int_H \nabla \tilde{w}(x', x_n) . \nabla v \, dx + \int_{\{x_n < 0\}} \nabla(\tilde{w}(x', -x_n)) . \nabla v \, dx$$

$$= - \int_H \tilde{g}(x', x_n) v \, dx + \int_{\{x_n < 0\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (\nabla_z \tilde{w})(x', \underbrace{-x_n}_{z}) . \nabla v(x', x_n) \, dx$$

Transformation

$$g(x', x_n) = (x', -x_n), \quad \frac{\partial g}{\partial x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \left|\det \frac{\partial g}{\partial x}\right| = 1$$

$$\int_{\{z>0\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \nabla \tilde{w}(x', z).(\nabla_{x_n} v)(x', \underbrace{-z}_{x_n})\, dx$$

$$= \int_{\{z>0\}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \nabla \tilde{w}(x', z). \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} \nabla_z(\underbrace{v(x', -z)}_{=:\hat{v}(x',z)})\, dx$$

$$= \int_{H=\{z>0\}} \nabla \tilde{w}(x', z).\nabla \hat{v}(x', z)\, dx$$

$$= - \int_{H=\{z>0\}} \tilde{g}(x', z)\hat{v}(x', z)\, dx = - \int_{\{x_n<0\}} \tilde{g}(x', -x_n)\, \underbrace{\hat{v}(x', -x_n)}_{=v(x',x_n)}\, dx$$

$$G(x', x_n) := \begin{cases} \tilde{g}(x', x_n) & , x_n > 0 \\ \tilde{g}(x', -x_n) & , x_n < 0 \end{cases}, \quad G \in L^2(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} \nabla W . \nabla v\, dx = - \int_{\mathbb{R}^n} G v\, dx, \quad \forall v \in H^1(\mathbb{R}^n) \supseteq \mathcal{D}(\mathbb{R}^n)$$

$\Delta W = G \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \Rightarrow \Delta W = G \quad \text{in } L^2(\mathbb{R}^n)$

Lemma: $W \in H^2(\mathbb{R}^n) \Rightarrow w = \varphi u \in H^2(\Omega)$

Case: $k \in \mathcal{D}$

$w := \varphi u$

$\tilde{w} \in H^1(H := \{x_n > 0\})$ ($\tilde{w}$ continuation of w by 0 outside Ω)

For $v \in \{v \in H^1(H) : \gamma_k v = 0 \text{ on } \Gamma_k\}$:

$$\int_H \nabla \tilde{w}.\nabla v \, dx = \int_\Omega \nabla(\varphi u).\nabla v \, dx = \underbrace{\int_\Omega u \nabla \varphi.\nabla v \, dx}_{I} + \underbrace{\int_\Omega \varphi \nabla u.\nabla v \, dx}_{II}$$

$$I = -\int_\Omega \nabla.(u\nabla\varphi) v \, dx + \underbrace{\sum_j \int_{\Gamma_j} \gamma_j u \gamma_j v \nabla\varphi.\nu_j \, d\sigma}_{=\int_{\Gamma_k} \gamma_k u \gamma_k v \nabla\varphi.\nu_k \, d\sigma = 0}$$

u variat. sol., $\forall v \in V := \{v \in H^1(\Omega) : \gamma_j v = 0 \text{ on } \Gamma_j, j \in \mathcal{D}\}$

$$\int_\Omega \nabla u . \nabla v \, dx = - \int_\Omega f v \, dx$$

$v \in \{v \in H^1(H) : \gamma_k v = 0 \text{ on } \Gamma_k\}$, $v' := \varphi v \in V$

$$\int_\Omega \nabla u . \nabla(\varphi v) \, dx = \int_\Omega v \nabla u . \nabla \varphi + \underbrace{\varphi \nabla u . \nabla v}_{II} \, dx = - \int_\Omega f \varphi v \, dx$$

$$(*) \quad \int_H \nabla \tilde{w} . \nabla v \, dx = - \int_\Omega \underbrace{(2 \nabla u . \nabla \varphi + u \Delta \varphi + f \varphi)}_{:= g \in L^2(\Omega), \quad \tilde{g} \in L^2(H)} v \, dx,$$

$$\forall v \in \{v \in H^1(H) : \gamma_k v = 0 \text{ on } \Gamma_k\}$$

Odd reflection through $\{x_n = 0\}$

$$W(x', x_n) := \begin{cases} \tilde{w}(x', x_n) & , x_n > 0 \\ -\tilde{w}(x', -x_n) & , x_n < 0 \end{cases}$$

$W \in H^1(\mathbb{R}^n)$?

$$\nabla W(x', x_n) := \begin{cases} \nabla \tilde{w}(x', x_n) & , x_n > 0 \\ -\nabla(\tilde{w}(x', -x_n)) & , x_n < 0 \end{cases}$$

$$\psi \in \mathcal{D}(\mathbb{R}^n)^3 : \quad \int_{\mathbb{R}^n} \nabla W(x', x_n).\psi \, dx = \dots$$

$$= -\int_{\mathbb{R}^n} W(x', x_n).\nabla \psi \, dx + \int_{\{x_n=0\}} (\gamma \tilde{w}(x', x_n) - (-\gamma \tilde{w}(x', -x_n)))\psi.\nu \, d\sigma$$

$$\underbrace{\gamma \tilde{w}(x', x_n)}_{=0} = -\underbrace{\gamma \tilde{w}(x', -x_n)}_{=0} \quad \text{on } x_n = 0 \quad \checkmark \quad (w = \varphi u)$$

For $v \in H^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \nabla W.\nabla v\, dx = \int_H \nabla \tilde{w}(x', x_n).\nabla v\, dx + \int_{\{x_n<0\}} -\nabla(\tilde{w}(x', -x_n)).\nabla v\, dx$$

$$= \int_{H=\{x_n>0\}} \nabla \tilde{w}(x', x_n).\nabla v(x', x_n)\, dx + \int_{H=\{z>0\}} -\nabla \tilde{w}(x', z).\nabla \hat{v}(x', z)\, dx$$

$$= \int_H \nabla \tilde{w}(x', x_n).\nabla(v(x', x_n) - \hat{v}(x', x_n))\, dx$$

$$v'(x', x_n) := v(x', x_n) - \underbrace{\hat{v}(x', x_n)}_{v(x', -x_n)} \in H^1(H)$$

$$\gamma v'(x', x_n) = 0 \quad \text{on } x_n = 0$$

$$= -\int_{\{x_n>0\}} \tilde{g}(x', x_n)(v(x', x_n) - \hat{v}(x', x_n))\, dx$$

$$= -\int_{\{x_n>0\}} \tilde{g}(x', -x_n) v(x', x_n) - \int_{\{x_n>0\}} -\tilde{g}(x', -x_n) \underbrace{\hat{v}(x', -x_n)}_{=v(x', x_n)}\, dx$$

$$G(x', x_n) := \begin{cases} \tilde{g}(x', x_n) & , x_n > 0 \\ -\tilde{g}(x', -x_n) & , x_n < 0 \end{cases}, \quad G \in L^2(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} \nabla W . \nabla v\, dx = -\int_{\mathbb{R}^n} G v\, dx, \quad \forall v \in H^1(\mathbb{R}^n) \supseteq \mathcal{D}(\mathbb{R}^n)$$

$\Delta W = G \quad$ in $\mathcal{D}'(\mathbb{R}^n) \Rightarrow \Delta W = G \quad$ in $L^2(\mathbb{R}^n)$

Lemma: $W \in H^2(\mathbb{R}^n) \Rightarrow w = \varphi u \in H^2(\Omega)$

Local regularity away from vertices and in 3d in addition edges:

The variational solution $u \in H^2(\Omega')$, $\forall$ open subset Ω' of Ω s.t. its closure $\overline{\Omega'}$ does not meet the vertices and in 3d in addition the edges.

Choose φ as plateau function:

- $\varphi = 1$ in Ω'
- $supp(\varphi) = \overline{\Omega'}$
- $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$

Neumann boundary conditions in 3d

Choose $\varphi \in \mathcal{D}(\bar{\Omega})$:
- whose support meets only interior points of Γ_k
- $\frac{\partial \varphi}{\partial \nu_k} = 0$ on Γ_k

Theorem: $w := \varphi u \in H^2(\Omega)$

$$\gamma_k\left(\frac{\partial(\varphi u)}{\partial \nu_k}\right) = \gamma_k\left(u\frac{\partial \varphi}{\partial \nu_k} + \varphi\frac{\partial u}{\partial \nu_k}\right) = \varphi\gamma_k\left(\frac{\partial u}{\partial \nu_k}\right) \quad \text{on } \Gamma_k$$

$$\int_\Omega \nabla w.\nabla v\, dx = -\int_\Omega gv\, dx, \quad \forall v \in V$$

$$\int_\Omega \nabla w.\nabla v\, dx = -\int_\Omega \Delta wv\, dx + \sum_j \int_{\Gamma_j} \gamma_j v \gamma_j\left(\frac{\partial(\varphi u)}{\partial \nu_j}\right) d\sigma, \quad \forall v \in V$$

$$= -\int_\Omega gv\, dx + \int_{\Gamma_k} \gamma_k v\, \varphi\gamma_k\left(\frac{\partial u}{\partial \nu_k}\right) d\sigma, \quad \forall v \in V$$

$$\int_{\Gamma_k} \gamma_k v \, \varphi \gamma_k \left(\frac{\partial u}{\partial \nu_k} \right) d\sigma = 0, \quad \forall v \in V$$

$k \in \mathcal{N} \Rightarrow \gamma_k v$ is an arbitrary function in $\tilde{H}^{\frac{1}{2}}(\Gamma_k)$

$$\varphi \gamma_k \left(\frac{\partial u}{\partial \nu_k} \right) = 0 \quad \text{in } \tilde{H}^{\frac{1}{2}}(\Gamma_k)$$

$$\gamma_k \left(\frac{\partial u}{\partial \nu_k} \right) = 0 \quad \text{in } H^{\frac{1}{2}}(\omega),$$

$\forall$ open set ω whose closure $\bar{\omega}$ is contained in Γ_k

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