Mesh Grading towards Singular Points
Seminar : Elliptic Problems on Non-smooth Domain

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Outline

Singularities in BVP

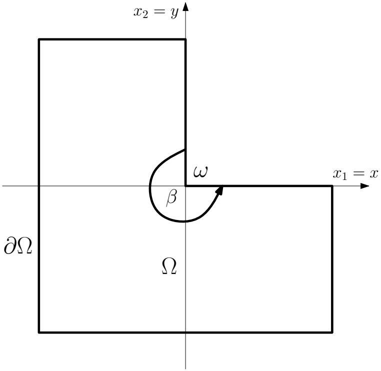
$$Lu \equiv -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + b_i\frac{\partial u}{\partial x_j} + \sigma u = f, \quad in \quad \Omega$$

$$u|_{S_1} = 0, \quad \left(a_{ij}\frac{\partial u}{\partial N} + \sigma u\right)|_{S_2} = 0$$

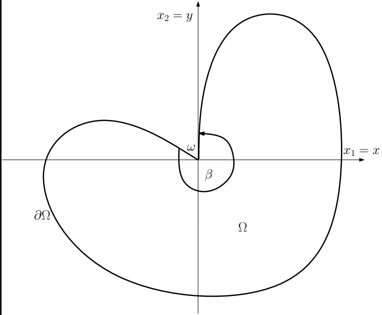
Cases of Singularities:

- Discontinuous coefficients (a_{ij}).
- Discontinuous right-hand side (f).
- Jump in boundary data.
- Domains with corner points.

Domains with corner points



L-shape



Heart

where

- $\beta > \pi$.
- Singularity at ω.

Recap...

$$-\Delta u + u = f \text{ in } \Omega \subset \mathbb{R}^2$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega := S$$

Find $u \in H = W_2^1(\Omega)$:

$$L_\Omega(u, \phi) = (f, \phi)_\Omega, \quad \forall \phi \in H$$

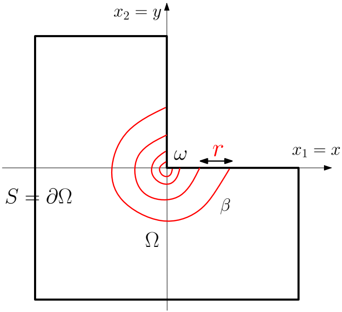
where

- $u = u_R + u_S$
  - u_R- Regular part.
  - $u_S = \sum_{0<\lambda<1} \gamma\psi(r, \theta)$ is Singular part.
- $\psi(r, \theta) = \xi(r) r^\lambda cos\lambda\theta$
  - $\lambda = \pi/\beta$ and $1/2 < \lambda < 1$.
  - $\xi(r)-$ smooth cut-off function.

Outline

Construction of Graded Mesh

Aim : To construct a mesh such that node distribution becomes denser towards the singular point.



let $h > 0$

$$r_i = (ih)^{1/\mu}, \quad i = 0, \dots, N$$
$$h_i = r_i - r_{i-1}, \quad i = 1, \dots, N$$

with $0 < \mu < 1$ and $N = [h^{-1}]$.

$$C_1(ih)^{\mu^{-1}-1} \leq \frac{h_i}{h} \leq C_2(ih)^{\mu^{-1}-1}$$

Example : 1D Graded Mesh

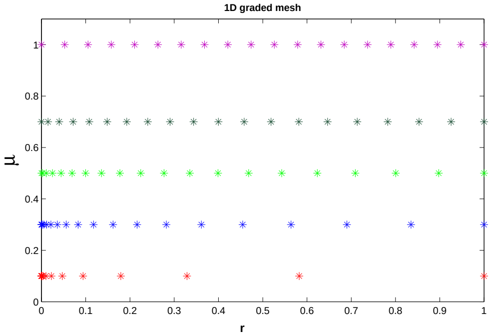
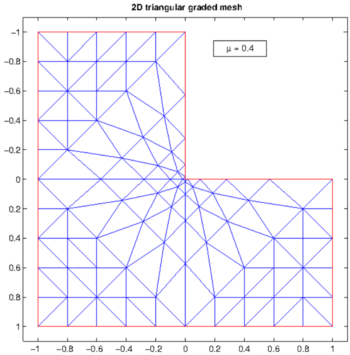


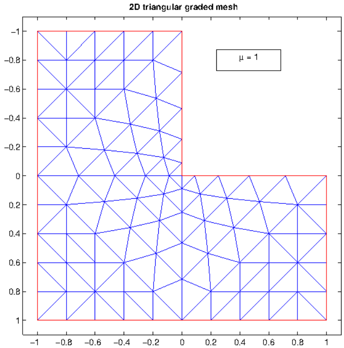
Figure : 1D graded mesh with varying μ

Example : L-Shape Graded Mesh

graded mesh with $\mu = 0.4$



No grading with $\mu = 1.0$



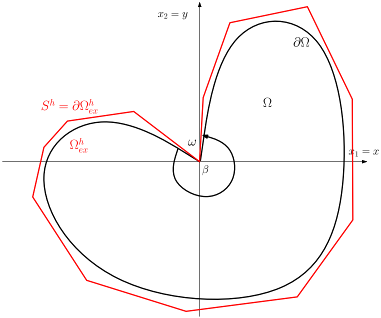
Example : Curved domain

$$\vartheta_i = \{r_{i-1} \leq r \leq r_i\}, \quad i = 1, \ldots, N$$

$$D = \bigcup_{i=1}^{N} \vartheta_i$$

- S^h is located exterior to Ω ($S^h \cap \Omega = \emptyset$)
  - nodes $x \in S^h \wedge z \in D$: $dist(x, S) \leq \delta h^2 = \mathcal{O}(h^2)$
  - nodes $x \in S^h \wedge z \in \vartheta_i$: $dist(x, S) \leq \delta h_i^2$

Domain with polygon

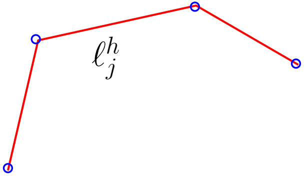


$S^h = \bigcup_{i=1}^{N} \ell_j^h$

- nodes
  $(\ell_j^h) \in \vartheta_i : |\ell_j^h| \geq \ell_0 h_i$
- $\ell_j^h \cap D = \emptyset : |\ell_j^h| \geq \ell_0 h$

where $\ell_0, \delta \neq C(h)$.

node description



Triangulation $(\mathcal{T}_h)$ of Ω_{ex}^h: Conditions to satisfy:

1. $\delta, \delta' \in \mathcal{T}_h : \delta \cap \delta' := \{\emptyset, \text{joint vertex}, \text{joint edge}\}$
   - $\overline{\delta} \cap \vartheta_i \neq \emptyset : \ell_1 h_i \leq \ell = |e| \leq \ell_2 h_i, \quad \forall e \in \partial\delta$
   - $\overline{\delta} \cap D = \emptyset : \ell_1 h \leq \ell = |e| \leq \ell_2 h, \quad \forall e \in \partial\delta$
2. $\theta_\delta \geq \theta_0 = const. > 0 \quad \forall \delta \in \mathcal{T}_h$
3. $N_i := |\{\delta \in \mathcal{T}_h : \overline{\delta} \cap \vartheta_i\}| \leq N_0 i$

where $\ell_1, \ell_2, N_0 \neq C(h)$.

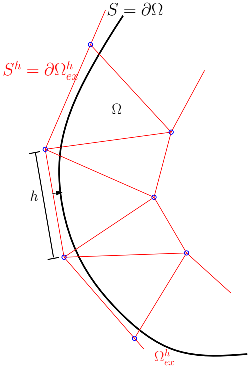


Figure : dist$(S, S^h) = \mathcal{O}(h^2)$

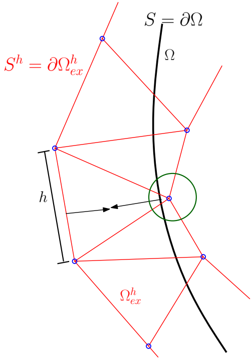


Figure : dist$(S, S^h) = \mathcal{O}(h)$

bad condition number in the case of natural BC (right figure).

Remark

- $\overline{R_h}$:= number of nodes $= \mathcal{O}(h^{-2})$.
- number of nodes not located in $D = \mathcal{O}(h^{-2})$
- number of nodes (M) located in D is :

$$M \leq C \sum_{i=1}^{N} N_i \leq C N_0 \sum_{i=1}^{N} i \leq C h^{-2} = \mathcal{O}(h^{-2}).$$

Outline

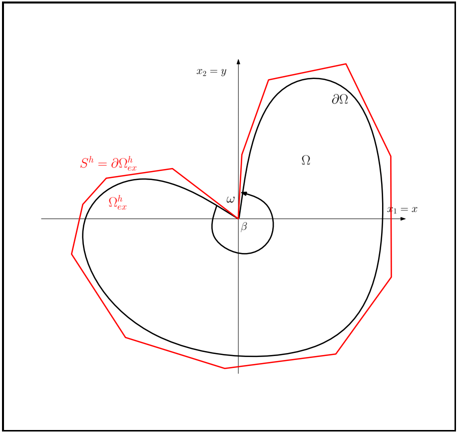
Singularities in BVP

Construction of Graded Mesh

FE-Scheme Error Estimates

Numerical Results

FE-Scheme Error Estimates



Find $u \in W_2^1(\Omega)$:

$$L_\Omega(u, \phi) = (f, \phi)_\Omega \quad \forall \phi \in H$$

Find
$\tilde{v} \in H_h = S^1(\Omega^h_{ex})$:

$$L_\Omega(\tilde{v}, \phi) = (f, \phi)_\Omega \quad \forall \phi \in H_h$$

Cea's Lemma : $\|u - \tilde{v}\|_{1,\Omega} \leq C \min_{\phi \in H_h} \|u - \phi\|_{1,\Omega}$

- $u = \gamma\psi + w \in H$ with $w \in W_2^2(\Omega) \subset W_2^2(\widetilde{\Omega})$

Interpolant :

- $\Pi_h : L_2 \longrightarrow H_h$
- $\tilde{u} = \Pi_h u = \gamma \Pi_h \psi + \Pi_h w = \gamma\tilde{\psi} + \tilde{w} \in H_h$

$$\|u - \tilde{v}\|_{1,\Omega} \leq C\|u - \tilde{u}\|_{1,\Omega_{ex}^h}$$

- For $\delta = \triangle \in \mathcal{T}_h : \psi = 0$, $i.e. u = w$

$$\|u - \tilde{u}\|_{1,\triangle}^2 = \|w - \tilde{w}\|_{1,\triangle}^2 \leq Ch^2\|w\|_{2,\square}^2$$

CASE I : $\triangle \cap [\vartheta_1 \cup \vartheta_2] \neq \emptyset$: Show

$$\|u - \tilde{u}\|^2_{1,\triangle} \leq C \left(h^{2\lambda/\mu} |\gamma|^2 + h^2 \|w\|^2_2 \right)$$

Ideas for Proof:

1. $r \leq Ch^{1/\mu}$
2. $\|\psi\|^2_{1,\triangle} \leq Ch^{2\lambda/\mu}$ and $\|\tilde{\psi}\|^2_{1,\triangle} \leq Ch^{2\lambda/\mu}$.
3. $\|w - \tilde{w}\|^2_{1,\triangle} \leq Ch^2 \|w\|^2_{2,\square}$
4. $\|u - \tilde{u}\|^2_{1,\triangle} \leq 4|\gamma|^2 \left(\|\psi\|^2_{1,\triangle} + \|\tilde{\psi}\|^2_{1,\triangle} \right) + 2\|w - \tilde{w}\|^2_{1,\triangle}$

CASE II : $\triangle \cap [\vartheta_1 \cup \vartheta_2] = \emptyset$:

$$\|u - \tilde{u}\|^2_{1,\Omega^h_{ex}} \leq C \left[h^{2\frac{\lambda}{\mu}} |\gamma|^2 \left(1 + \sum_{i=1}^{N} i^{2\frac{\lambda}{\mu} - 4} \right) + h^2 \|w\|^2_{2,\Omega} \right]$$

Ideas for Proof:

1. $|D^2\psi|^2 \leq Cr^{2\lambda-4}$
2. $\|w - \tilde{w}\|^2_{1,\triangle} \leq Ch^2 \|w\|^2_{2,\square}$
3. $\|u - \tilde{u}\|^2_{1,\triangle} \leq C \left(h_i^4 r_i^{2\lambda-4} |\gamma|^2 + h_i^2 \|w\|^2_{2,\square} \right)$
4. sum over all triangles $\triangle \in \mathcal{T}_h$

- For $\mu = 1$

$$\sum_{i=1}^{N} i^{2\lambda-4} \leq C < \infty, \quad \forall N.$$

$$\|u - \tilde{v}\|_{1,\Omega_{ex}^h} \leq Ch^{\lambda} \|f\|_{0,\Omega}$$

- For $\mu < \lambda$, i.e. $\frac{\lambda}{\mu} > 1$

$$\sum_{i=1}^{N} i^{2\frac{\lambda}{\mu}-4} \leq Ch^{2-2\frac{\lambda}{\mu}}$$

$$\|u - \tilde{v}\|_{1,\Omega_{ex}^h} \leq Ch \|f\|_{0,\Omega}$$

Outline

Construction of Graded Mesh: IGA

- Example : L-Shape with 2 patches

$$\begin{cases} -\Delta u = 0 & in \ \Omega \\ u = g_D & on \, \partial\Omega \end{cases}$$

$u(x, y) = (x^2 + y^2)^{1/3} \sin((2 \arctan(y/x) + \pi)/3)$.

- $\|u - u_h\|_{L^2} = \mathcal{O}(h^{4/3})$.
- $\|u - u_h\|_{H^1} = \mathcal{O}(h^{2/3})$.

Knot Vector

- $\Xi_{1,2} = \{0, 0, 1, 1\}$.
- $p = 1$ (bilinear FEM).

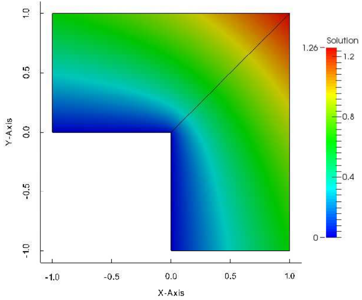
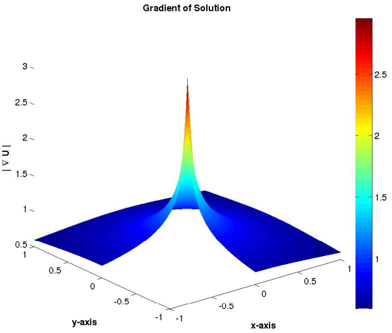
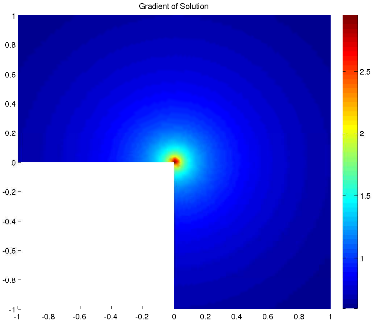


Figure : L-Shape solution on multi-patch.

- Gradient of the solution.





Ref: [http://www.math.uci.edu/chenlong/226/Ch4AFEM.pdf]

- **Knot Vector Grading:** insert knots closer to singularity.

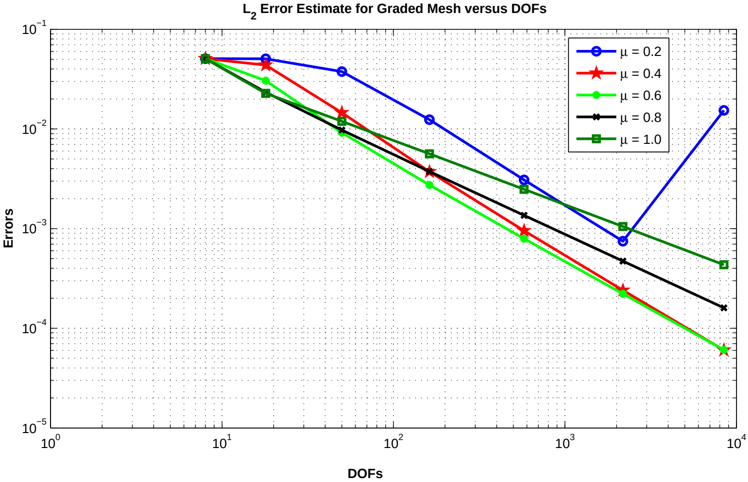


Figure : L_2-Errors of graded mesh plotted against DOFs.

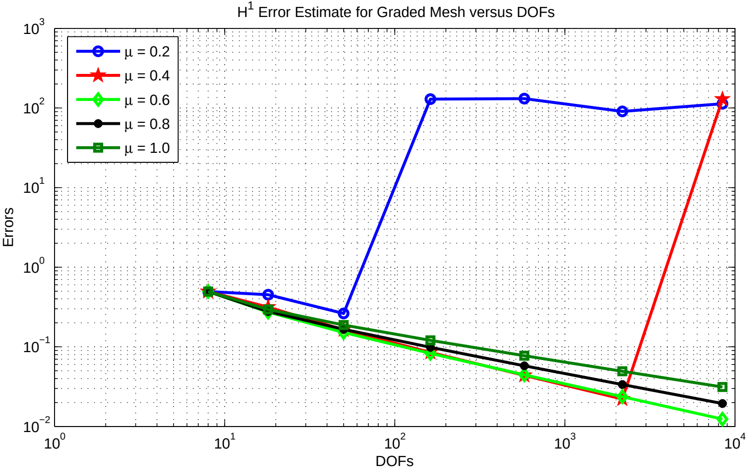


Figure : H^1-Errors of graded mesh plotted against DOFs.

Conclusion

1. For $\mu = 1$, the convergence rate for $p \geq 1$ is
   - $\|u - u_h\|_{L^2} = \mathcal{O}(h^{4/3})$.
   - $\|u - u_h\|_{H^1} = \mathcal{O}(h^{2/3})$.
2. Knot grading with $0 < \mu < 0.6$
   - $\|u - u_h\|_{L^2} = \mathcal{O}(h^2)$.
   - $\|u - u_h\|_{H^1} = \mathcal{O}(h)$.
3. Remarks on the artifacts in errors:
   - condition number of matrix deteriorates for $\mu \to 0$.
   - related solver issues.
   - computation of the H^1-norm close to the singularity.