Edge Behaviour in 3D
Seminar : Elliptic Problems on Non-smooth Domain

Stephen Edward Moore

Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences, Linz, Austria

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Recap...

- Variational Solution (see talk by K. Rafetseder):

Classical Formulation

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \text{ on } \Gamma_j,\ j \in \mathcal{D} \\ \frac{\partial u}{\partial \nu_j} &= 0 \text{ on } \Gamma_j,\ j \in \mathcal{N} \end{aligned}$$

Variational Formulation

Find $u \in V$:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv \, dx, \quad \forall v \in V$$

- where

$$V := \{u \in H^1(\Omega); \gamma_j u = 0 \; on \; \Gamma_j, j \in \mathcal{D}\}$$

- Classical Formulation $\Longleftrightarrow$ Variational Formulation
  - $\Rightarrow$ ✓
  - $\Leftarrow$ ×

Theorem

Let Ω be a polygonal domain in $\mathbb{R}^2$; if $u \in H^1(\Omega)$ fulfills conditions :

- $\Delta u = f$ *in* $L^2(\Omega)$.
- $\gamma_j u = 0$ *in* $\widetilde{H}^{1/2}(\Gamma_j),\ j \in \mathcal{D}$
- $\gamma_j(\frac{\partial u}{\partial \nu_j}) = 0$ *in* $\widetilde{H}^{-1/2}(\Gamma_j),\ j \in \mathcal{N}$

then $\exists! u \in V$ *of*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv \, dx, \quad \forall v \in V$$

- $\Leftarrow$ ✓

Lemma

Let $u \in H^1(\mathbb{R}^n)$ *and* $\Delta u \in L^2(\mathbb{R}^n)$ *then* $u \in H^2(\mathbb{R}^n)$.

Theorem (Interior Regularity)

Let $u \in V$ *be the variational solution of*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv \, dx, \quad \forall v \in V$$

then $\varphi u \in H^2(\Omega), \forall \varphi \in \mathcal{D}(\Omega)$.

Theorem (Local regularity up boundary)

Let $u \in V$ *be the variational solution of*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv \, dx, \quad \forall v \in V$$

then $\varphi u \in H^2(\Omega), \forall \varphi \in \mathcal{D}(\overline{\Omega})$ *whose support intersects* Γ *only at interior points of* Γ_j.

Recap: Fredholm Property

$$\Delta : V^2 \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

- Complete characterization of N (see talk by W. Krendl):

if $v \in N$ then $v \in D(\Delta; L^2(\Omega))$,

$$\begin{aligned} \Delta v &= 0 \\ \gamma_j v &= 0,\ j \in \mathcal{D} \\ \gamma_j \left(\partial v \partial \nu_j\right) &= 0,\ j \in \mathcal{N} \end{aligned}$$

- $v \in N \Longrightarrow v \in C^\infty(\overline{\Omega} \setminus \cup_j \bigcup_\delta(S_j))$

Fredholm Property

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

- Corner behaviour (see talk by P. Gangl)

- $N = \mathcal{R}(\Delta; V^2)^{\perp}$
- Augment space V^2 by $X \subset H^s(\Omega)$ with $s \in (1, 2)$ s.t Δ is surjective onto $L^2(\Omega)$
  Then : $f \in L^2(\Omega) \implies u \in H^s(\Omega)$
  - $u = u_R + u_S$

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

- **$v \in N$ near Dirichlet corners (S_j)**

- Transition to polar coordinates about S_j; $(r, \theta) = (x_1, x_2)$.
- Ansatz : $v(r, \theta) = r^\lambda \varphi(\theta) \Longrightarrow -\varphi''(\theta) = -\lambda^2 \varphi(\theta)$
- Expansion of $v \in N, \forall (r, \theta) \in (0, \rho) \times (0, \omega_j)$

$$v(r, \theta) = \sum_{m \geq 1} \alpha_m r^{\lambda_{j,m}} \varphi_{j,m}(\theta) + \sum_{0 < \lambda_{j,m} < 1} \beta_m r^{-\lambda_{j,m}} \varphi_{j,m}(\theta)$$

and

$$|\alpha_m| \leq L\sqrt{m}\rho^{-\lambda_{j,m}}, \quad \alpha_m, \beta_m \in \mathbb{R}$$

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

- **Match expansions:**
  For each j and each $\lambda_{j,m} \in (0,1), \exists\, \sigma_{j,m} \in N$:

$$\sigma_{j,m} - \eta_j r_j^{-\lambda_{j,m}} \varphi_{j,m}(\theta) \in H^1(\Omega)$$

- $dim(N) = \sum_j \text{card}\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$

$$j \in \mathcal{D} \text{ and } j+1 \in \mathcal{D} \left(\lambda_{j,m} = \frac{m\pi}{\omega_j}\right) : \begin{cases} 0 \ \textit{if } \omega_j \leq \pi \\ 1 \ \textit{if } \omega_j > \pi \end{cases}$$

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

Derivation of Singular solution :

- $\Delta(V^2 + X) = L^2(\Omega)$
- $X \subset H^s(\Omega), 1 < s < 2$
- Let $S_{j,m}(r_j, \theta_j) = \eta_j(r_j) r_j^{\lambda_{j,m}} \varphi_{j,m}(\theta_j)$ and $F_{j,m}$ as its Laplacian :

$$F_{j,m} := \Delta S_{j,m} \in (\overline{\mathcal{D}})$$

- $S_{j,m}$ is the variational solution to

$$\begin{aligned} \Delta S_{j,m} &= F_{j,m} \\ \gamma_k S_{j,m} &= 0 \ \mathit{for}\ k \in \mathcal{D} \\ \gamma_k \left(\partial S_{j,m}/\partial \nu_k\right) &= 0 \ \mathit{for}\ k \in \mathcal{N} \end{aligned}$$

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

$S_{j,m}(r_j, \theta_j) = \eta_j(r_j) r_j^{\lambda_{j,m}} \varphi_{j,m}(\theta_j)$ satisfies

- $S_{j,m} \in H^1(\Omega)$ for $\lambda_{j,m} \geq 0$
- $S_{j,m} \notin H^2(\Omega)$ for $0 < \lambda_{j,m} < 1$
- $\{S_{j,m}\}$ and $\{\Delta S_{j,m}\}$ linearly independent.

$F_{j,m}$ is not orthogonal to N if $\lambda_{j,m} < 1$.

$$\Delta : V^2 + X \to \mathcal{R}(\Delta; V^2) + N = L^2(\Omega)$$

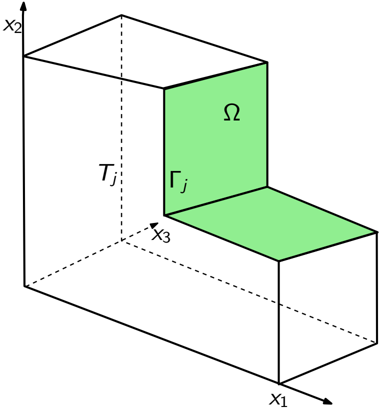
- For $f \in L^2(\Omega), \exists! u \in V$ of variational problem $\forall v \in V$ and $\exists! c_{j,m}$:

$$u_R := u - \sum_j \left(\sum_{0<\lambda_{j,m}<1} c_{j,m} S_{j,m} \right) \in H^2(\Omega)$$

- For each corner S_j : $u \in H^s(\bigcup_\delta(S_j))$ for every $s \leq 2$: $s < 1 + \inf_m\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$
- Globally: $u \in H^s(\Omega)$ for every $s \leq 2$: $s < 1 + \inf_{j,m}\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}$

Classical Formulation

$$\Delta u = f \text{ in } Q$$
$$u = 0 \text{ on } T_j, j \in \mathcal{D}$$
$$\frac{\partial u}{\partial \nu_j} = 0 \text{ on } T_j, j \in \mathcal{N}$$



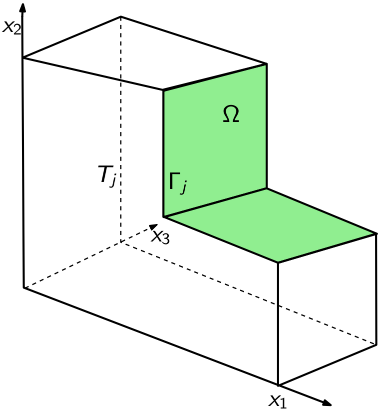
where

$$Q = \Omega \times \mathbb{R}, \quad \Omega \subset \mathbb{R}^2$$
$$T_j = \Gamma_j \times \mathbb{R}$$

Variational Formulation

Given $f \in L^2(Q), \exists! u \in W$

$$\int_Q \nabla u \cdot \nabla v \, dx = -\int_Q fv \, dx, \quad \forall v \in W$$



where

$$W := \{u \in H^1(Q); \gamma_j u = 0 \; on \; T_j, j \in \mathcal{D}\}$$

Classical Formulation

$$\Delta u = f \text{ in } Q$$
$$u = 0 \text{ on } T_j,\ \ j \in \mathcal{D}$$
$$\frac{\partial u}{\partial \nu_j} = 0 \text{ on } T_j,\ \ j \in \mathcal{N}$$

Variational Formulation

Find $u \in W$:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega fv \, dx, \quad \forall v \in W$$

- where
$$W := \{u \in H^1(\Omega); \gamma_j u = 0 \; on \; T_j, j \in \mathcal{D}\}$$
- Classical Formulation $\Longleftrightarrow$ Variational Formulation ✓

Fredholm Property

$$\Delta(\Longrightarrow \Delta - \xi^2 I) : V^2 + X \to \mathcal{R}(\Delta; V^2) + N(\Longrightarrow N\xi) = L^2(\Omega)$$

- Corner behaviour (see talk by P. Gangl) $\Longrightarrow$ **Edge behaviour**

- $N\xi = \mathcal{R}(\Delta - \xi^2 I; V^2)^\perp$
- Augment space V^2 by $X \subset H^s(\Omega)$ with $s \in (1,2)$ s.t Δ is surjective onto $L^2(\Omega)$
  Then : $f \in L^2(\Omega) \Longrightarrow u \in H^s(\Omega)$
  - $u = u_R + u_S$

Fourier Transform

Definition

if $u \in L_1(\mathbb{R}^n)$, then Fourier transform $\hat{u}$ is defined as

$$\hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx, \quad y \in \mathbb{R}^n$$

Properties

Assume $u, v \in L_2(\mathbb{R}^n)$. Then

- $\int_{\mathbb{R}^n} u\overline{v}\, dx = \int_{\mathbb{R}^n} \hat{u}\overline{\hat{v}}\, dy.$
- $\widehat{(D^\alpha u)} = (iy)^\alpha \hat{u}, \quad \forall\alpha$ *such that* $D^\alpha u \in L^2(\mathbb{R}^n)$.
- *if* $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, *then* $\widehat{(u * v)} = (2\pi)^{n/2}\hat{u}\hat{v}.$
- $u = \widehat{(\hat{u})}$

$$\psi \in H^s(\mathbb{R}^n) \Longleftrightarrow (1 + |\xi|^2)^{s/2}\hat{\psi}(\xi) \in L_2(\mathbb{R}^n)$$

Consider Fourier transform w.r.t x_3

$$\hat{u}(x', \xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_3} u(x', x_3)\, dx_3, \quad x' = (x_1, x_2)$$

Applying Fourier tranform to BVP yields

$$\begin{aligned} \Delta' \hat{u} - \xi^2 \hat{u} &= \hat{f} \text{ in } \Omega,\ \dot{\forall} \xi \in \mathbb{R} \\ \hat{u} &= 0 \text{ on } \Gamma_j,\ j \in \mathcal{D} \\ \frac{\partial \hat{u}}{\partial \nu_j} &= 0 \text{ on } \Gamma_j,\ j \in \mathcal{N} \end{aligned}$$

where

- $\Delta' := \Delta_{x'}$
- $\mathcal{F}_{x_3} : \Delta' u + \frac{\partial^2 u}{\partial x_3^2} - f = 0.$

3D to 2D

Theorem

Let $u \in W$ be the variational solution of

$$\int_Q \nabla u \cdot \nabla v \, dx = - \int_Q fv \, dx, \quad \forall v \in W.$$

Then for almost every $\xi \in \mathbb{R}$, $\hat{u} \in V$ is the variational solution of

$$\int_\Omega \nabla' \hat{u} \cdot \nabla' w \, dx' + \xi^2 \int_\Omega \hat{u} w \, dx = - \int_\Omega \hat{f} w \, dx', \quad \forall w \in V.$$

where

- $\nabla' := \nabla_{x'}$

Edge Behaviour

$$\Delta - \xi^2 I : V^2 + X \to \mathcal{R}(\Delta - \xi^2 I; V^2) + N\xi = L^2(\Omega)$$

- For $f \in L^2(\Omega), \exists! \hat{u} \in V$ of variational problem $\forall v \in V$ and $\exists! c_{j,m}(\xi)$:
- $\hat{u}(x', \xi) := \hat{u}_R(x', \xi) + \hat{u}_S(x', \xi)$
- $\hat{u}_R(x', \xi) := \hat{u}(x', \xi) - \sum_j \left(\sum_{0<\lambda_{j,m}<1} c_{j,m}(\xi) e^{-\xi r_j} S_{j,m}(x') \right)$
  - $u_R = \mathcal{F}_{x_3}^{-1}(\hat{u_R}) \in H^2(Q)$
  - $u_S = \mathcal{F}_{x_3}^{-1}(\hat{u_S}) = \sum_j \sum_{0<\lambda_{j,m}<1} \left[\mathcal{F}_{x_3}^{-1} \left(c_{j,m}(\xi) e^{-\xi r_j} \right) \right] S_{j,m}(x')$
  - $\implies u_R = u - \sum_j \sum_{0<\lambda_{j,m}<1} [\cdots] \, S_{j,m}(x')$

Theorem (Basic result)

There exists a constant K such that

$$\|\hat{u}_R\|_{2,\Omega} + \xi\|\hat{u}_R\|_{1,\Omega} + \xi^2\|\hat{u}_R\|_{0,\Omega} \leq K\|\hat{f}\|_{0,\Omega}$$

and that

$$|c_{j,m}(\xi)| \leq K(1+\xi)^{\lambda_{j,m}-1}\|f\|_{0,\Omega}$$

Proof :

- P. Grisvard, Singularities in Boundary Value Problems. Masson (1992), pp. 60 - 72.

Theorem (main results)

For $f \in L^2(Q)$, there exists a unique $u \in W$ (variational) solution of

$$\int_Q \nabla u \cdot \nabla v \, dx = \int_Q fv \, dx, \quad \forall v \in W.$$

and there exists a unique functions $\psi_{j,m} \in H^{1-\lambda_{j,m}}(\mathbb{R})$ of the variable x_3 such that

$$u - \sum_j \{ \sum_{0<\lambda_{j,m}<1} (K * \psi_{j,m}) S_{j,m} \} \in H^2(Q)$$

where $K = r/\pi(r^2 + x_3^2)$ and the $*$ denotes the convolution in the variable x_3.

where

- $K * \psi_{j,m} = (r/\pi) \int_{\mathbb{R}} \psi_{j,m}(x_3 - t)(r^2 + t^2)^{-1} \, dt$

and $r^2 = x_1^2 + x_2^2$

- **Proof :** BLACKBOARD or see P. Grisvard, Singularities in Boundary Value Problems (pp. 60 - 72.)

Remark

In the case of a pure Dirichlet problem $(\mathcal{N} = \emptyset)$ or of a pure Neumann problem $(\mathcal{D} = \emptyset)$, only the non-convex edges contribute and with a single term corresponding to $m = 1$.

$$u - \sum_{\omega_j > \pi} (K * \psi_{j,1}) S_{j,1} \in H^2(Q)$$

where $\psi_{j,1} \in H^{1-\pi/\omega_j}(\mathbb{R})$.

Corollary

Given j, let V_j be an open neighborhood of S_j in $\overline{\Omega}$ which does not contain any other corner, then $u \in H^s(V_j \times \mathbb{R})$ for every $s \leq 2$ such that $s < \lambda_{j,m} + 1, \forall \lambda_{j,m}$ such that $0 < \lambda_{j,m} < 1$.

- **Proof :** P. Grisvard (1992).

Remark

f is inifinitely differentiable in x_3 with values in $L^2(\Omega)$. Then u is inifinitely differentiable in x_3 with values in $H^s(V_j)$ where $s \leq 2$ such that $s < \lambda_{j,m} + 1, \forall \lambda_{j,m}$.