Part 2: Multigrid Methods for the computation of singular solutions and stress intensity factors

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Numeric Seminar, Linz, WS 2013

Used literature

S. C. Brenner, *Multigrid Methods for the computation of singular solutions and stress intensity factors : Corner singularities I.*, Department of Mathematics and Center for Computation and Technology, Mathematics of computation, April 1999, Volume 86, Number 226, Pages 559-583.

Introduction

Let be:

$\{\mathcal{T}_k\},\ k \geq 1$, a family of triangulations of Ω, where a regular subdivision $\mathcal{T}_{k+1}$ of is obtained from $\mathcal{T}_k$ by connecting the edges of the triangles in $\mathcal{T}_k$.

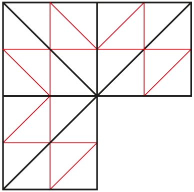


Figure: Triangulation

$V_k = \{v \in H_0^1(\Omega) :\ v|_T \in \mathcal{P}_1\ \forall T \in \mathcal{T}_k\}$ ... piecewise linear finite elements associated with $\mathcal{T}_k$.

Introduction

The discrete inner product $(\cdot,\cdot)_k$ defined by

$$(v_1, v_2)_k = h_k^2 \sum_{\text{vertices } p \text{ of } \mathcal{T}_k} v_1(p) v_2(p) \quad \forall v_1, v_2 \in V_k.$$

$\Rightarrow$ $(v, v)_k$ is spectral equivalent to $\|v\|^2_{L^2(\Omega)} \quad \forall v \in V_k.$

The operators $A_k : V_k \to V_k$ and $I_k^{k-1} : V_k \to V_{k-1}$ (Restriction Operator), defined by:

$$\begin{aligned}(A_k v_1, v_2)_k &= \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \quad \forall v_1, v_2 \in V_k \subset V_{k-1},\\ (I_k^{k-1} v, w)_{k-1} &= (v, w)_k \qquad \forall v \in V_k, w \in V_{k-1}.\end{aligned}$$

$\Rightarrow$ A_k symmetric, positive definite and the spectralradius $\rho(A_k) \lesssim h_k^{-2}$.

Standard k-th level multigrid iteration

The k-**th level multigrid iteration** with initial guess z_0 yields $MG(k, z_0, g)$ as an approximate solution to the equation

$$A_k z = g.$$

For $k = 1$, $MG(1, z_0, g)$ is the solution obtained from an exact solver, i.e. $MG(1, z_0, g) = A_1^{-1} g$.

For $k > 1$, there are two steps.

Smoothing Step: Let $z_l \in V_k$ $(1 \leq l \leq m)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\gamma_k}(g - A_k z_{l-1}), \quad 1 \leq l \leq m, \quad \text{(Richardson Relaxation)}$$

where $m \in \mathbb{N}_0$ independent of k, and $\gamma_k = C h_k^{-2}$ dominates $\rho(A_k)$.

Correction Step: Let $\overline{g} = I_k^{k-1}(g - A_k z_m) \in V_{k-1}$ and $q_i \in V_{k-1}$ $\big(0 \leq i \leq p,\ p = 1$ (V-cycle) or $p = 2$ (W-cycle)$\big)$ be defined recursively by

$$q_0 = 0 \quad \text{and} \quad q_i = \underbrace{MG(k-1, q_{i-1}, \overline{g})}_{\text{approx. of } A_{k-1}^{-1}\overline{g}}, \quad 1 \leq i \leq p.$$

The output is obtained by combining the two steps:

$$MG(k, z_0, g) = z_m + q_p.$$

Full multigrid algorithm 1

If $f \in L^2(\Omega)$, we use the nested iteration to compute κ_k and w_k.

The **nested iteration:**

For $k = 1$,

$$w_1 = A_1^{-1} g_1, \quad \text{where} \quad (g_1, v)_1 = \int_\Omega f\, v\, dx \quad \forall v \in V_1.$$

We set

$$\kappa_1 = 0 \quad \text{and} \quad u_1 = w_1.$$

For $k \geq 2$, $\kappa_k \in \mathbb{R}$ are computed by

$$\kappa_k = \frac{1}{\pi} \left(\int_\Omega f\, s_-\, dx + \int_\Omega u_{k-1} \Delta s_-\, dx \right),$$

and $w_k \in V_k$ is obtained recursively by

$$w_{k,0} = w_{k-1}, \qquad w_{k,l} = MG(k, w_{k,l-1}, g_k), \qquad w_k = w_{k,n} \quad \text{for } 1 \leq l \leq n,$$

where n is a positive integer independent of k, and $g_k \in V_k$ is defined by

$$(g_k, v)_k = \int_\Omega (fv\, dx + \kappa_k \Delta s_+)\, dx \quad \forall v \in V_k.$$

We define then u_k by

$$u_k = \kappa_k s_+ + w_k.$$

Full multigrid algorithm 2

If $f \in H^1(\Omega)$, we use the nested iteration to compute $\kappa_{\ell,k}$, $\ell \in \mathcal{L}$ and w_k.

The **nested iteration:**

For $k = 1$,

$$w_1 = A_1^{-1} g_1, \quad \text{where} \quad (g_1, v)_1 = \int_\Omega f\, v\, dx \quad \forall v \in V_1.$$

We set

$$\kappa_{\ell,1} = 0 \quad \text{for} \ell \in \mathcal{L}, \quad \text{and} \quad u_1 = w_1.$$

For $k \geq 2$, $\kappa_{\ell,k} \in \mathbb{R}$ are computed by

$$\kappa_k = \frac{1}{\ell\pi}\left(\int_\Omega f\, s_{+,-\ell}\, dx + \int_\Omega u_{k-1} \Delta s_{+,-\ell}\, dx\right) \quad \text{for} \quad \ell \in \mathcal{L},$$

and $w_k \in V_k$ is obtained recursively by

$$w_{k,0} = \mathcal{J}_{k-1}^k w_{k-1}, \qquad w_{k,l} = MG(k, w_{k,l-1}, g_k), \qquad w_k = w_{k,n} \quad \text{for } 1 \leq l \leq n,$$

where n is a positive integer independent of k, and $g_k \in V_k$ is defined by

$$(g_k, v)_k = \int_\Omega \left(fv\, dx + \sum_{\ell \in \mathcal{L}} \kappa_{\ell,k} \Delta s_{+,\ell} \right) dx \quad \forall v \in V_k.$$

We define then u_k by

$$u_k = \sum_{\ell \in \mathcal{L}} \kappa_{\ell,k} s_{+,\ell} + w_k.$$

The intergrid transfer operator $\mathcal{J}_{k-1}^{k}$

For

$$Q_k \subset H_0^1(\Omega))(k = 0, 1, 2, \dots) \text{ quadratic Lagrange finite element space associated with } \mathcal{T}_k. \qquad (1)$$

we define the interpolation operators

$$\begin{aligned} \mathcal{I}_{k-2}^{k-1} : Q_{k-2} &\to V_{k-1} \\ w &\to v, \text{ such that } v(p) = w(p) \quad \forall \text{ vertices } p\text{of } \mathcal{T}_{k-1} \end{aligned}$$

which is an isomorphism and

$$\begin{aligned} \mathcal{I}_{k-2}^{k} : Q_{k-2} &\to V_{k} \\ w &\to v, \text{ such that } v(p) = w(p) \quad \forall \text{ vertices } p \text{ of } \mathcal{T}_{k}, \end{aligned}$$

and further the **intergrid transfer operator**:

$$\mathcal{J}_{k-1}^{k} = \mathcal{I}_{k-2}^{k} \circ (\mathcal{I}_{k-2}^{k-1})^{-1} : V_{k-1} \to V_k \quad \text{for } k = 2, 3, \dots.$$

Contraction properties for the k-th level iteration

Convergence result for the k-th level iteration in the energy norm:

Lemma

Let $p = 1$ (V-cycle) or $p = 2$ (W-cycle) and $m \geq 1$ in the k-th level iteration. Then there exists a $\delta < 1$, independent of k, such that

$$|z - MG(k, z_0, g)|_{H^1(\Omega)} \leq \delta |z - z_0|_{H^1(\Omega)}. \tag{2}$$

Convergence result for the k-th level iteration in the $\| \cdot \|_{H^{1-(\pi/\omega)+\epsilon}(\Omega)}$ norm:

Theorem

Let $p = 2$ (W-cycle), $0 < \delta < 1$, $0 < \epsilon < \pi/\omega$ and $\alpha_\epsilon = 1 - \pi/\omega \neq 1/2$. If the number of smoothing steps m in the k-th level iteration is sufficiently large, then we have

$$\|z - MG(k, z_0, g)\|_{H^{\alpha_\epsilon}(\Omega)} \leq \delta \|z - z_0\|_{H^{\alpha_\epsilon}(\Omega)}. \tag{3}$$

Convergence Analysis for the full multigrid algorithm 1

Theorem

Let $p = 2$ (W-cycle), $0 < \epsilon < \pi/\omega$, $\alpha_\epsilon = 1 - \pi/\omega \neq 1/2$ and the number of smoothing steps m in the k-th level iteration be sufficiently large, that (2) and (3), hold for $0 < \delta < 1$. If the number of nested iterations n is sufficiently large, then we have

$$|w - w_k|_{H^1(\Omega)} \lesssim h_k \|f\|_{L^2(\Omega)}, \tag{4}$$

$$|\kappa - \kappa_k| \lesssim_\epsilon h_k^{1+\pi/\omega-\epsilon} \|f\|_{L^2(\Omega)}, \tag{5}$$

$$\|w - w_k\|_{H^{\alpha_\epsilon}(\Omega)} \lesssim_\epsilon h_k^{1+\pi/\omega-\epsilon} \|f\|_{L^2(\Omega)}. \tag{6}$$

where w_k and κ_k are computed by **Full multigrid algorithm 1**.

Corollary

Under the assumption of theorem 3, we have

$$|u - u_k|_{H^1(\Omega)} \lesssim h_k \|f\|_{L^2(\Omega)}. \tag{7}$$

Convergence Analysis for the full multigrid algorithm 2

Properties of the intergrid transfer operator $\mathcal{J}_{k-1}^{k} : V_{k-1} \to V_k$

Lemma

We have the following estimates for $\mathcal{J}_{k-1}^{k}$:

$$|\mathcal{J}_{k-1}^{k} v|_{H^1(\Omega)} \lesssim h_k |v|_{H^1(\Omega)} \quad \forall v \in V_{k-1}, \tag{8}$$

$$|\Pi_k \eta - \mathcal{J}_{k-1}^{k} \Pi_{k-1} \eta|_{H^1(\Omega)} \lesssim h_k^{1+t} \|\eta\|_{H^{2+t}(\Omega)} \quad \forall v \in V_{k-1}, \tag{9}$$

where

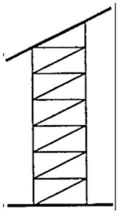
$$\begin{aligned} \Pi_k : H^1(\Omega) &\to V_k \\ w &\to v, \ \textit{such that } v(p) = w(p) \quad \forall \textit{ vertices } p \textit{ of } \mathcal{T}_k, \end{aligned}$$

is the nodal interpolation operator associated with V_k and $0 \leq t \leq 1$.

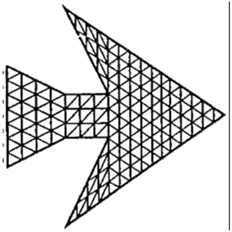
Uniform band condition (UBC)

Definition

A uniform band in a triangulation is a collection of triangle between two parallel lines, such that any two triangle s sharing a common side form a parallelogram (see figure 2). We say a triangulation satisfies the uniform band condition (UBC), if it can be divided completely into uniform bands (see figure 3).



Figure



Figure

Remark

One can always find a triangulation satisfying the uniform band condition for any polygonal domain whose vertices all have rational coordinates, and the uniform band condition is preserved by regular subdivision.

Super convergence result

We define the Ritz projection operator $P_k : H^2(\Omega) \to V_k$ by

$$\int_\Omega \nabla(\eta - P_k\eta) \cdot \nabla v \, dx = 0 \quad \forall \eta \in H_0^1(\Omega),\ v \in V_k.$$

Lemma

Suppose the triangulations $\mathcal{T}_k$ satisfy the uniform band condition and $\eta \in H^3(\Omega) \cap H_0^1(\Omega)$. Then

$$|\Pi_k\eta - P_k\eta|_{H^1(\Omega)} \lesssim h_k^2 \|\eta\|_{H^3(\Omega)}.$$

Corollary

Suppose the triangulations $\mathcal{T}_k$ satisfy the uniform band condition and $\eta \in H^3(\Omega) \cap H_0^1(\Omega)$ for $0 \leq t \leq 1$. Then

$$|\Pi_k\eta - P_k\eta|_{H^1(\Omega)} \lesssim h_k^{1+t} \|\eta\|_{H^{2+t}(\Omega)}.$$

Convergence Analysis for the full multigrid algorithm 2

Using this previous **superconvergence result** we can show:

Theorem

Let $f \in H^1(\Omega)$. Assume that the triangulations $\mathcal{T}_k$ satisfy the uniform band condition, $p = 1$ (V-cycle), or $p = 2$ (W-cycle), and $m \geq 1$. If the number of nested iterations n is sufficiently large, then we have

$$|\Pi_k w - w_k|_{H^1(\Omega)} \lesssim_\epsilon h_k^{2-\epsilon} \|f\|_{H^1(\Omega)}, \tag{10}$$

$$\sum_{\ell \in \mathcal{L}} |\kappa_\ell - \kappa_{\ell,k}| \lesssim_\epsilon h_k^{2-\epsilon} \|f\|_{H^1(\Omega)}, \tag{11}$$

where $\mathcal{L} = \{\ell \in \mathbb{N} : \ \ell\pi/\omega < 2\}$ and w_k, $\kappa_{l,k}$ are computed by **Full multigrid algorithm 2**.

Corollary

Under the assumption of theorem 7, we have

$$|u - u_k|_{H^1(\Omega)} \lesssim h_k \|f\|_{H^1(\Omega)} \quad \text{and} \quad \max_p |u(p) - u_k(p)|_{H^1(\Omega)} \lesssim_\epsilon h_k^{2-\epsilon} \|f\|_{H^1(\Omega)}, \tag{12}$$

where the maximum is taken over all the vertices $p \in \mathcal{T}_k$.

Remark

If for all internal angles ω of Ω we have $\ell\omega \neq \pi/2$ for all $\ell \in \mathbb{N}$, then

$$w \in H^3(\Omega),$$

from which follows the ϵ-independent estimates:

$$|\Pi_k w - w_k|_{H^1(\Omega)} \lesssim h_k^2 \|f\|_{H^1(\Omega)},$$

$$\sum_{\ell \in \mathcal{L}} |\kappa_\ell - \kappa_{\ell,k}| \lesssim h_k^2 \|f\|_{H^1(\Omega)},$$

$$\max_p |u(p) - u_k(p)|_{H^1(\Omega)} \lesssim h_k^2 |\ln h_k|^{1/2} \|f\|_{H^1(\Omega)}.$$

Model data:

Domain Ω:
Γ-shaped domain (see figures 4 and 5) with vertices $(0,0),(0,1),(1,1),(-1,1),(-1,-1)$ and $(0,-1)$.

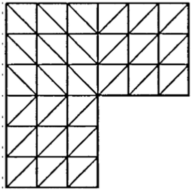


Figure: Γ-shape triangulation (without UBC)

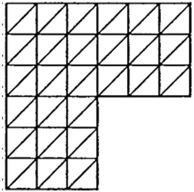


Figure: Γ-shape triangulation (with UBC)

Model data:

Finite element:

P1-Lagrange finite element.

Meshsize:

The meshsize h_k for the k-th level grid is taken by 2^{-k}.

Multigrid parameters:

Using a W-cycle k-th (p=2) level iteration, with 5 smoothing steps (m=n=5). Why $m = n = 5$? Because the numerical results do not appear to improve for any larger m or n.

Singular function on the Γ-shaped domain:

$$s_1(r,\theta) = \eta(r) r^{2/3} \sin(2/3\theta),$$
$$s_2(r,\theta) = \eta(r) r^{4/3} \sin(4/3\theta).$$

Cut-off function η:

$$\eta(r) = \begin{cases} 1 & 0 \le r \le \frac{1}{4} \\ -192r^5 + 480r^4 - 440r^3 + 180r^2 - \frac{135r}{4} + \frac{27}{8}, & \frac{1}{4} \le r \le \frac{3}{4}, \\ 0 & 3/4 \le r. \end{cases}$$

(see figure 6).

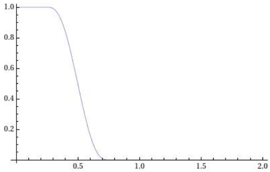


Figure: Cut off function η

Input data:

We will now compute a solution of the Poisson equation

$$\begin{aligned} -\Delta u =& f \quad \text{in } \Omega, \\ u =& 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{13}$$

using

- Standard full multigrid algorithm.
- Full mulitgrid algorithm 1.
- Full mulitgrid algorithm 2.

for

$$f = -\Delta s_1 - \Delta s_2 + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2),$$

with exact solution

$$u = \underbrace{s_1}_{\in H^1(\Omega)\ \wedge\ \notin H^2(\Omega)} + \underbrace{s_2}_{\in H^2(\Omega)\ \wedge\ \notin H^{3-\epsilon}(\Omega)} + (x - x^3)(y^2 - y^4) \ \text{(exact solution)}.$$

Experiment 1: Standard full multigrid algorithm (SFA)

Solving the Poisson equation (13)

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

with

$$f = -\Delta s_1 - \Delta s_2 + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$$

and exact solution

$$u = s_1 + s_2 + (x - x^3)(y^2 - y^4),$$

by the standard full multigrid algorithm, on the the Γ-shape, using the discretization fulfills not the uniform band condition presented in figure 4.

Approximations for the stress intensity factors κ_k are computed by the extraction formula

$$\kappa_h = \frac{1}{\pi}\left(\int_\Omega f s_- \, dx + \int_\Omega u \Delta s_- \, dx\right),$$

using the P1 finite element solution u_k obtained by standard full multigrid algorithm.

Experiment 1: Standard full multigrid algorithm (SFA)

$$e_k = |\Pi_k u - u_k|_{H^1(\Omega)} \ \ldots \text{error in the energy norm},$$

$$\sigma_k = \log_2\left(\frac{|\kappa_{k-1} - 1|}{|\kappa_k - 1|}\right) \ \ldots \text{convergence rate for stress intensity factor},$$

$$\epsilon_k = \log_2\left(\frac{e_{k-1}}{e_k}\right) \ \ldots \text{convergence rate in the energy norm}.$$

$\Rightarrow$ Theoretical: $\sigma_k = \mathcal{O}(h_k^{4/3})$ and $\epsilon_k = \mathcal{O}(h_k^{2/3})$

k	κ_k	σ_k	e_k	ϵ_k
1	1.6999229601	–	1.27093×10^{0}	–
2	1.2589102299	1.43	5.91072×10^{-1}	1.1045
3	1.1036407706	1.32	1.61387×10^{-1}	1.8728
4	1.0287080790	1.85	5.74371×10^{-2}	1.4905
5	1.0073492045	1.97	2.76732×10^{-2}	1.0535
6	1.0020544785	1.84	1.64752×10^{-2}	0.7482
7	1.0005531037	1.89	1.02811×10^{-2}	0.6803
8	1.0001571227	1.82	6.46930×10^{-2}	0.6683
9	1.0000458701	1.78	4.07502×10^{-3}	0.6668
10	1.0000142397	1.69	2.56715×10^{-3}	0.6666
11	1.0000046460	1.62	1.61722×10^{-3}	0.6666

Figure: Results for SFA.

Experiment 2: Full multigrid algorithm 1 (FMGA1)

Solving the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

with

$$f = -\Delta s_1 - \Delta s_2 + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$$

and exact solution

$$u = s_1 + s_2 + (x - x^3)(y^2 - y^4),$$

by the standard full multigrid algorithm 2, on the the Γ-shape, using the discretization fulfills not the uniform band condition presented in figure 4.

We compute κ_k and $w_k \in V_k$ which are approximations of ...

- *stress intensity factor $\kappa = 1$,*
- *and the regular part of the exact solution $w = s_2 + (x - x^3)(y^2 - y^4)$.*

Experiment 2: Full multigrid algorithm 1 (FMGA1)

$$e_k = |\Pi_k w - w_k|_{H^1(\Omega)} \ \dots \text{error in the energy norm},$$

$$\sigma_k = \log_2\left(\frac{|\kappa_{k-1} - 1|}{|\kappa_k - 1|}\right) \ \dots \text{convergence rate for stress intensity factor},$$

$$\epsilon_k = \log_2\left(\frac{e_{k-1}}{e_k}\right) \ \dots \text{convergence rate in the energy norm}.$$

$\Rightarrow$ Theoretical: $\sigma_k = \mathcal{O}(h_k^{5/3})$ and $\epsilon_k = \mathcal{O}(h_k)$

k	κ_k	σ_k	e_k	ϵ_k
1	–	–	7.929×10^{-1}	–
2	1.69992296014	–	8.364×10^{-1}	–0.07
3	0.82132136706	1.97	2.322×10^{-1}	1.85
4	1.02037630458	3.13	3.456×10^{-2}	2.75
5	0.99943755129	5.18	6.236×10^{-3}	2.47
6	1.00003984026	3.82	1.595×10^{-3}	1.97
7	1.00000536058	2.89	4.200×10^{-4}	1.93
8	1.00000234005	1.20	1.170×10^{-4}	1.84
9	1.00000057569	2.02	3.567×10^{-5}	1.71
10	1.00000012632	2.19	1.204×10^{-5}	1.57
11	1.00000002876	2.13	4.397×10^{-6}	1.45
12	1.00000000746	1.95	–	–

Figure: Results for FMGA1.

Experiment 3: Full multigrid algorithm 2 (FMGA2)

Solving the Poisson equation (13)

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

with

$$f = -\Delta s_1 - \Delta s_2 + 6x(y^2 - y^4) + (x - x^3)(12y^2 - 2)$$

and exact solution

$$u = s_1 + s_2 + (x - x^3)(y^2 - y^4),$$

by the standard full multigrid algorithm 2, on the the Γ-shape, using the discretization fulfills the uniform band condition presented in figure 5.

We compute $\kappa_{1,k}, \kappa_{2,k}$ and $w_k \in V_k$ which are approximations of ...

- *stress intensity factors $\kappa_1 = \kappa_2 = 1$,*
- *and the regular part of the exact solution $w = (x - x^3)(y^2 - y^4)$.*

Experiment 3: Full multigrid algorithm 2 (FMGA2)

$$e_k = |\Pi_k w - w_k|_{H^1(\Omega)} \ldots \text{error in the energy norm},$$

$$\sigma_{i,k} = \log_2\left(\frac{|\kappa_{i,k-1} - 1|}{|\kappa_{i,k} - 1|}\right) \ldots \text{convergence rate for stress intensity factor},$$

$$\epsilon_k = \log_2\left(\frac{e_{k-1}}{e_k}\right) \ldots \text{convergence rate in the energy norm}.$$

$\Rightarrow$ Theoretical: $\sigma_{i,k} = \mathcal{O}(h_k^2)$ and $\epsilon_k = \mathcal{O}(h_k^2)$

k	$\kappa_{1,k}$	$\sigma_{1,k}$	$\kappa_{2,k}$	$\sigma_{2,k}$	e_k	ϵ_k
1	–	–	–	–	1.124×10^{0}	–
2	1.6229151283	–	1.17131298888	–	7.361×10^{-1}	0.61
3	0.8859991798	2.45	0.99336080108	4.69	1.417×10^{-1}	2.38
4	1.0091773397	3.63	1.00029662538	4.48	1.131×10^{-2}	3.65
5	0.9999856171	9.32	1.00023130682	0.36	5.829×10^{-4}	4.28
6	1.0000653041	−2.18	1.00002651087	3.13	1.551×10^{-4}	1.91
7	1.0000136298	2.26	1.00000976600	1.44	3.636×10^{-5}	2.09
8	1.0000044994	1.60	1.00000116447	3.07	9.574×10^{-6}	1.93
9	1.0000011279	2.00	1.00000029598	1.98	2.376×10^{-6}	2.01
10	1.0000002659	2.08	1.00000008791	1.75	5.810×10^{-7}	2.03
11	1.0000000638	2.06	1.00000002475	1.82	1.433×10^{-7}	2.02
12	1.0000000163	1.97	1.00000000585	2.08	–	–

Figure: Results for FMA2.

Conclusions

- The multigrid methods use the simplest finite element.
- Since the grid are generated by connecting midpoints, it is easy to parallelize the algorithms.
- For more regular f, there exists a singular function representations where the regular part w is also more regular. In such cases multigrid methods with higher orders of convergence can be developed using higher order elements.
- Note that other superconvergence results which are less restrictive that the one based on the "uniform band" condition can also be used if the are available.