Fredholm Property of the Δ-operator in 2-dimensional polygonal domain

Wolfgang Krendl[1]

[1]Doctoral Program Computational Mathematics, Johannes Kepler University, Linz, Austria

Numeric Seminar, Linz, WS 2013

Used literature

P. Grisward, *Singularities in boundary value problems*, University of Nice (France) ,Masson Springer, Berlin (1992).

Pages: 43 - 49.

Basic a-priori inequalities in polygons

Theorem

For every $v \in V^2(\Omega)$ where

$$V^2 = \{v \in H^2(\Omega) : \quad \gamma_j(v) = 0 \text{ for } j \in \mathcal{D} \text{ and } \gamma_j(\partial v/\partial \nu_j) = 0 \text{ for } j \in \mathcal{N}\}.$$

the identity

$$||\Delta u||_{0,\Omega}^2 = ||D_1^2 u||_{0,\Omega}^2 + ||D_1^2 u||_{0,\Omega}^2 + 2||D_1 D_2 u||_{0,\Omega}^2$$

holds.

For the proof of this theorem we use the following lemma:

Lemma

The identity

$$\int_\Omega D_1^2 u \, D_2^2 u \, \mathrm{d}x = \int_\Omega (D_1 D_2 u)^2 \, \mathrm{d}x$$

holds for all $u \in V^2(\Omega)$.

Basic a-priori inequalities in polygons

Theorem

Assume that Ω is a bounded polygonal open subset of $\mathbb{R}^2$ and that $\mathcal{D}$, is not empty. Then there exists a constant $C(\Omega)$ such that:

$$\|u\|_{2,\Omega} \leq C(\Omega)\|\Delta u\|_{0,\Omega}, \tag{1}$$

for every $u \in V^2(\Omega)$.

Fredholm property in 2d

Let us consider the operator

$$\Delta u : V^2(\Omega) \to L^2(\Omega).$$

The inequality

$$\|u\|_{2,\Omega} \le C(\Omega)\|\Delta u\|_{0,\Omega},$$

proved in the previous Theorem already shows that Δ is injective and has a closed range. The question is now:

How is the range $\mathcal{R}(\Delta)$ of the Δ-operator completely identified?

To answer this question, it is enough to identify its orthogonal,

$$\mathcal{R}(\Delta)^\perp = \left\{ v \in L^2(\Omega) : \int_\Omega v \Delta u \, \mathrm{d}x = 0 \text{ for all } u \in V^2(\Omega) \right\},$$

since

$$L^2(\Omega) = \mathcal{R}(\Delta) + \mathcal{R}(\Delta)^\perp.$$

Characterization of $\mathcal{R}(\Delta)^\perp$

Notation: For positive s we denote by $\tilde{H}^s(\Gamma_j)$ the space of all u defined in Γ_j such that $\tilde{u} \in H^s(\mathbb{R})$, where $\tilde{u}$ is the continuation of u by zero outside Γ_j.

Lemma

Let be $v \in \mathcal{R}(\Delta)^\perp$. Then v belongs to

$$D(\Delta, L^2(\Omega)) = \{v \in L^2(\Omega) : \ \Delta v \in L^2(\Omega)\}$$

and is solution of the adjoint boundary value problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \Omega, \\ \gamma_j(v) &= 0 \quad \text{in } \tilde{H}^{3/2}(\Gamma_j) \quad \text{for } j \in \mathcal{D}, \\ \gamma_j(\partial v/\partial \nu_j) &= 0 \quad \text{in } \tilde{H}^{1/2}(\Gamma_j) \quad \text{for } j \in \mathcal{N}. \end{aligned}$$

Characterization of $\mathcal{R}(\Delta)^\perp$

Notation:

- $\mathcal{M}'$ ... set of all $j \in \mathcal{N}$ such that $j+1 \in \mathcal{D}$, and the angle ω_j is either 90° or 270° degrees.
- $\mathcal{M}''$ ... set of all $j \in \mathcal{D}$ such that $j+1 \in \mathcal{N}$ and the angle ω_j is either 90° or 270° degrees.

Lemma

Every $v \in \mathcal{R}(\Delta)^\perp$ satisfies:

$$\begin{aligned} \int_\Omega v \Delta \eta_j \,\mathrm{d}x &= 0 \quad \forall j \in \mathcal{N}^2, \\ \int_\Omega v \Delta (y_j \eta_j) \,\mathrm{d}x &= 0 \quad \forall j \in \mathcal{M}', \\ \int_\Omega v \Delta (x_j \eta_j) \,\mathrm{d}x &= 0 \quad \forall j \in \mathcal{M}''. \end{aligned} \tag{2}$$

Characterization of $\mathcal{R}(\Delta)^\perp$

Theorem

Let $v \in D(\Delta, L^2(\Omega))$ be such that

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \Omega, \\ \gamma_j(v) &= 0 \quad \text{in } \tilde{H}^{3/2}(\Gamma_j) \quad \text{for } j \in \mathcal{D}, \\ \gamma_j(\partial v/\partial \nu_j) &= 0 \quad \text{in } \tilde{H}^{1/2}(\Gamma_j) \quad \text{for } j \in \mathcal{N}. \end{aligned}$$

and assume in addition that v fulfills the conditions (2), then $v \in \mathcal{R}(\Delta)^\perp$.

Characterization of $\mathcal{R}(\Delta)^\perp$

Lemma

Let be $v \in \mathcal{R}(\Delta)^\perp$, then $v \in C^\infty(\overline{\Omega} \setminus V)$ where V is any neighborhood of the corners S_j.