# Zenger Corrections

Johannes Höller

21.1.2014

# Content

# Table of Contents

# Energy Functional

## Problem definition

The problem we are interested in is: Find $u \in V$:

$$\begin{aligned} a(u,v) &= \langle f, v\rangle \forall v \in V \\ u|_{\partial\Omega} &= 0 \end{aligned} \tag{1}$$

where $a(u,v) := \int_\Omega (\nabla u)(\nabla v)dx$ and without loss of generality $\Omega = \{(x,y) \in \mathbb{R}^2 | x < 0 \vee y < 0\}$ is an L-shape domain with the non-convex corner in the origin. $V = H_0^1(\Omega)$ or alternatively $V = V_h$.

## Definition:

The functional $E(u) := \frac{1}{2}a(u,u) - \langle f, u\rangle$ is called energy functional.

# Numerical problems

- We know that for the given problem there is a singular part to the solution
- Ignoring this in numerical treatment is bad
- Zenger proposed to counteract by changing the problem just a little
- consider a modified bilinear form

$$a_\gamma(u_h, v_h) = a(u_h, v_h) - \underbrace{\gamma K(u_h, v_h)}_{=:K_\gamma(u_h,v_h)} \tag{2}$$

for the discretized problem.

## Localized Correction

The correction $K_\gamma(u_h, v_h)$ should be local, that is:

$$K_\gamma(u_h, v_h) = \gamma \sum_{\{i,j\} \in N(0,0)} \beta_{ij} u_h(x_i) v_h(x_j) \tag{3}$$

# General Properties

Let $u$ be the solution to (1) and $R_h u$ the solution of the corresponding (unmodified) discrete problem. Then we know:

- Equivalent minimization problem:

$$a(u, v) = \langle f, v\rangle \ \forall v \in V \Leftrightarrow u \text{ minimizes } E(u) \qquad (4)$$

- Galerkin Orthogonality:

$$a(u - R_h u, v_h) = 0 \ \forall v_h \in V_h \qquad (5)$$

- Error minimization: $R_h u$ minimizes $\min_{v_h \in V_h} a(u - v_h, u - v_h)$
- the energy of the error = error of energy:

$$a(u - R_h u, u - R_h u) = a(u, u) - a(R_h u, R_h u) \qquad (6)$$

no proofs

# Properties for the modified problem

- equivalent minimization works for $a_\gamma$ as well
- modified orthogonality:

$$a(u - R_{h,\gamma}u, v_h) + K_\gamma(R_{h,\gamma}u_h, v_h) = 0 \ \forall v_h \in V_h \tag{7}$$

- modified Error minimization: $R_{h,\gamma}u$ minimizes $\min_{v_h \in V_h} a(u - v_h, u - v_h) - K_\gamma(v_h, v_h)$
- the energy of the error = error of energy (modified):

$$a(u - R_{h,\gamma}u, u - R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) = a(u,u) - a_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) \tag{8}$$

# proofs

## modified orthogonality

We have:

$$a(u, v_h) = \langle f, v_h \rangle \ \forall v_h \in V_h$$
$$a(R_{h,\gamma}u, v_h) - K_\gamma(R_{h,\gamma}u, v_h) = \langle f, v_h \rangle \ \forall v_h \in V_h$$

taking the difference yields the claimed result:

$$a(u - R_{h,\gamma}u, v_h) + K_\gamma(R_{h,\gamma}u, v_h) = 0 \ \forall v_h \in V_h$$

# proofs

## modified error minimum

$$
\begin{aligned}
& a(u - R_{h,\gamma}u - v_h, u - R_{h,\gamma}u - v_h) - K_\gamma(R_{h,\gamma}u + v_h, R_{h,\gamma}u + v_h) \\
= \; & a(u - R_{h,\gamma}u, u - R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) \\
- \; & \underbrace{\left(a(u - R_{h,\gamma}u, v_h) + K_\gamma(R_{h,\gamma}u, v_h)\right)}_{=0} \underbrace{-a(v_h, R_{h,\gamma}u) - K_\gamma(v_h, R_{h,\gamma}u)}_{=0} \\
+ \; & \underbrace{a(v_h, v_h) - K_\gamma(v_h, v_h)}_{\geq 0}
\end{aligned}
$$

By ensuring that the modified bilinear form $a_\gamma$ is positive definite we have the requested result.

# proofs

## Energy of error = error of energy

$$
\begin{aligned}
& a(u - R_{h,\gamma}u, u - R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) \\
= \; & a(u, u - R_{h,\gamma}u) \underbrace{- a(R_{h,\gamma}u, u - R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u)}_{=0} \\
= \; & a(u,u) - a(u, R_{h,\gamma}u) = a(u,u) - a(R_{h,\gamma}u, R_{h,\gamma}u) \\
- \; & a(u - R_{h,\gamma}u, R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) + K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u) \\
= \; & a(u,u) \underbrace{- a(R_{h,\gamma}u, R_{h,\gamma}u) - K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u)}_{=-a_\gamma} \\
- \; & \underbrace{a(u - R_{h,\gamma}u, R_{h,\gamma}u) + K_\gamma(R_{h,\gamma}u, R_{h,\gamma}u)}_{=0} \\
= \; & a(u,u) - a_\gamma(R_{h,\gamma}u, R_{h,\gamma}u)
\end{aligned}
$$

# Table of Contents

# Regularized $\delta$-functions

It is common knowledge, that there is no function (in $L^2$) such that:

$$\int_\Omega f(x)\delta(x)dx = f(0)\forall f \in L^2(\Omega) \tag{9}$$

However it is possible that a function fulfills this in a finite dimensional subspace like for example linear polynomials.

$$\int_\Omega p(x)\delta(x)dx = f(0)\forall p \in \mathbb{P}^1(\Omega) \tag{10}$$

# Construction of $\delta_x$

- Let $\hat{K}$ be any Element of some triangular mesh.
- Choose $\omega \geq 0 \in C^\infty(\mathbb{R}^2)$ with $supp(\omega) \subseteq \hat{K}$
- without loss of generality we may assume $\int_{\hat{K}} \omega dx = 1$
- Now choose a basis $\{\varphi_0, \varphi_1, \varphi_2\}$ of $\mathbb{P}^1 = span\{1, x, y\}$ which is orthonormal with respect to the scalar product $(f, g) = \int_{\hat{K}} f(x)g(x)\omega(x)dx$
- We have: $\delta_x(y) = \sum_{i=0}^2 \varphi_i(x)\varphi_i(y)\omega(y)$

## proof

Let $v \in \mathbb{P}^1$ be $v(x) = \sum_{i=0}^2 a_i\varphi_i(x)$

$$\begin{aligned} \int_{\hat{K}} v(y)\delta_x(y)dy &= \sum_{i=0}^2 \varphi_i(x) \sum_{j=0}^2 a_j \overbrace{\int_{\hat{K}} \varphi_i(y)\varphi_j(y)\omega(y)dy}^{=\delta_{ij}} \\ &= \sum_{i=0}^2 a_i\varphi_i(x) = v(x) \end{aligned}$$

# Table of Contents

# Point-Error-estimates

We want to obtain an error estimate of the form $|u(x) - R_{h,\gamma}u(x)| \leq ???$, however this is not in $P^1$ so we can't use $\delta_x$ directly. For any point $x$ let $K_x \in T_h$ be the triangle containing $x$ (for $x$ sufficiently far from the singularity).

$$\begin{aligned} u(x) & - R_{h,\gamma}u(x) - \int_{K_x}(u(y) - R_{h,\gamma}u(y))\delta_x(y)dy = \int_{K_x}(u(x) - u(y))\delta_x(y)dy \\ &= \int_{K_x}\left(u'(x)(y-x) + \frac{1}{2}(y-x)u''(\xi(x,y))(y-x)\right)\delta_x(y)dy \\ &= u'(x)\underbrace{\int_{K_x}(y-x)\delta_x(y)dy}_{=0} \\ &+ \frac{1}{2}\underbrace{\int_{K_x}\delta_x(y)}_{=O(1)}\underbrace{(y-x)}_{O(h)}\underbrace{u''(\xi(x,y))}_{=O(1)}\underbrace{(y-x)}_{=O(h)}dy = O(h^2) \end{aligned}$$

# reformulate by bilinear form

**Now consider the dual problem, with right hand side $\delta_x$**

Find $g_x$ such that:

$$\begin{aligned} a(v, g_x) &= \langle \delta_x, v \rangle \forall v \in V \\ g_x &= 0 \text{ on } \partial\Omega \end{aligned}$$

This problem has a unique solution by Lax-Milgram because $\delta_x$ is in $L^2$ and $a(\cdot, \cdot)$ should fulfill appropriate conditions from the original problem. $g_x$ is called regularized greens function.

By choosing the primary problems error as a test function we get:

$$a(u - R_{h,\gamma} u, g_x) = \langle \delta_x, u - R_{h,\gamma} u \rangle = u(x) - R_{h,\gamma} u(x) + O(h^2) \quad (11)$$

That means, that with the regularized greens-function the point error is representable by the bilinear form.

Thank you for your attention!