Boundaries and Traces

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Definition 1

Let $\Omega \subseteq \mathbb{R}^n$ open. The boundary Γ of Ω is called continuous respectively Lipschitz if for every $x \in \Gamma$ there exists a neighborhood V of x in $\mathbb{R}^n$ and a new set of orthogonal coordinates such that:

- V is a hypercube in the new coordinates: $V = \{(y_1, ..., y_n)| - a_i < y_i < a_i, 1 \leq i \leq n\}$
- There is a continuous resp. Lipschitz function ϕ defined on $V' = \{(y_1, ..., y_{n-1})| - a_i < y_i < a_i, 1 \leq i \leq n-1\}$ and :
  - $|\phi(y')| \leq a_n/2$ for every $y' \in V'$
  - $\Omega \cap V = \{y \in V | y_n \leq \phi(y')\}$
  - $\Gamma \cap V = \{y \in V | y_n = \phi(y')\}$

Definition 2

Let $\Omega \subseteq \mathbb{R}^n$ open. $\bar{\Omega}$ is called a continuous resp. Lipschitz submanifold with boundary in $\mathbb{R}^{n-1}$, if for every $x \in \Gamma$ there is a neighborhood V of x in $\mathbb{R}^n$ and a mapping ψ from V to $\mathbb{R}^n$ such that:

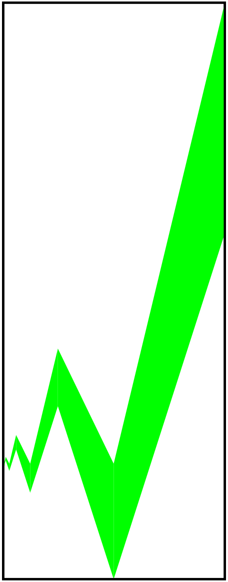
- ψ is injective
- ψ and ψ^{-1} (defined on $\psi(V)$) are continuous resp. Lipschitz
- $\Omega \cap V = \{y \in V | \psi_n(y) < 0\}$

Equivalent Definitions?

- Having two definitions commonly used one has to ask the question, what their differences are or whether they are maybe equivalent
- Consider: $\psi(y) = \{y_1, ..., y_{n-1}, y_n - \phi(y')\}$
- Therefore Definiton 1 $\implies$ Definition 2
- If everything is at least continuously differentiable one can use the implicit function theorem to get $\phi(\psi)$
  Then Definition 2 $\implies$ Definition 1.
- For only Lipschitz boundaries the latter does not hold

Counterexample

- Ω has infinitely many oscillations towards the origin
- It has a boundary with Lipschitz property according to Definition 2 by construction (not proved here)
- But it has no Lipschitz boundary according to Definition 1
- Any line segment starting at the origin will cut Γ infinitely often or never



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Sobolev Spaces

Definition $H^m(\Omega)$

For any integer $m \geq 0$ and $\Omega \subseteq \mathbb{R}^n$, we define $H^m(\Omega)$ as the space of all distributions u from Ω to $\mathbb{R}^n$ such that $D^\alpha u \in L_2$ for $|\alpha| \leq m$.

Norm $||.||_{m,\Omega}$

$$||u||^2_{m,\Omega} = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^2 dx \qquad (1)$$

Example: H^1

$$||u||^2_{1,\mathbb{R}^2} = ||u||^2_2 + \left|\left|\frac{\partial u}{\partial x}\right|\right|^2_2 + \left|\left|\frac{\partial u}{\partial y}\right|\right|^2_2 \qquad (2)$$

and for non-integers

Definition $H^s(\Omega)$

For non-integer $s > 0$ we define $H^s(\Omega)$ as the space of all distributions u from Ω to $\mathbb{R}^n$ such that:

- $s = m + \sigma$, m integer, $\sigma \in (0,1)$
- $u \in H^m(\Omega)$
- $\int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dxdy < +\infty$ for $|\alpha| = m$

Norm $||.||_{s,\Omega}$

$$||u||_{s,\Omega}^2 = ||u||_{m,\Omega}^2 + \sum_{|\alpha|=m} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dxdy$$

Basic Theorems

Sobolev's Theorem

For $k < s - n/2$ one has:

$$H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n).$$

Trace Theorem for Hyperplane

Define: $\gamma u(x_1, ..., x_{n-1}, x_n) = u(x_1, ..., x_{n-1}, 0)$.
The mapping $u \to (\gamma u, \gamma D_n u, ..., \gamma D_n^k u)$ defined for smooth u has for $k < s - 1/2$ a unique continuous extension as an operator from $H^s(\mathbb{R}^n)$ **onto** $\prod_{p=0}^k H^{s-p-1/2}(\mathbb{R}^{n-1})$.

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Trace Theorem for quadrant

Let $\Omega = \{(x,y) \in \mathbb{R}^2 | x > 0, y > 0\}$. The mapping:

$$\begin{aligned} u &\rightarrow \{f_0, f_1, g_0, g_1\} \\ f_0 &= u|_{y=0}, f_1 = \frac{\partial u}{\partial y}|_{y=0} \\ g_0 &= u|_{x=0}, g_1 = \frac{\partial u}{\partial x}|_{x=0} \end{aligned}$$

defined for smooth u has a unique continuous extension from $H^2(\Omega)$ **onto** the subspace of

$$T = H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+) \times H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+)$$

defined by:

- a $f_0(0) = g_0(0)$
- b_1 $\int_0^{+\infty} |\frac{\partial f_0}{\partial x}(t) - g_1(t)|^2/t dt < +\infty$
- b_2 $\int_0^{+\infty} |f_1(t) - \frac{\partial g_0}{\partial y}(t)|^2/t dt < +\infty$

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Special Case

At first consider the subspace E of $H^2(\Omega)$ with $g_0 = g_1 = 0$. Then $u \in E$ is equivalent to $\tilde{u} \in H^2(\mathbb{R} \times \mathbb{R}_+)$, where $\tilde{u}$ is the continuation of u by zero for $x < 0$.

Special case

The mapping $u \to \{f_0, f_1\}$ has a unique continuous extension from E **onto** the subspace of $H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+)$ defined by:

- a $f_0(0) = 0$
- b_1 $\int_0^{+\infty} |\frac{\partial f_0}{\partial x}(t)|^2/t dt < +\infty$
- b_2 $\int_0^{+\infty} |f_1(t)|^2/t dt < +\infty$

Proof of necessity

- Trace Theorem on hyperplanes for $\tilde{u}$:
  $\tilde{f}_0 \in H^{3/2}(\mathbb{R}), \tilde{f}_1 \in H^{1/2}(\mathbb{R})$.
- Thus $f_0 \in H^{3/2}(\mathbb{R}_+), f_1 \in H^{1/2}(\mathbb{R}_+)$ and $f_0(0) = 0$ (a) since $\tilde{f}_0$ is continuous by Sobolev's Theorem.
- For b_1, b_2 consider that $\tilde{f}_1$ and $\frac{\partial \tilde{f}_0}{\partial x}$ are $\in H^{1/2}(\mathbb{R})$.
- Thus

$$\int_{\mathbb{R}^2} |\tilde{f}_1(t) - \tilde{f}_1(s)|^2/|t-s|^2 dtds < +\infty$$

  by the $H^{1/2}$ norm.
- Restrict integration to $t > 0, s < 0$:

$$\begin{aligned} & \int_0^{+\infty} \int_{-\infty}^0 |f_1(t)|^2/|t-s|^2 dtds \\ = & \int_0^{+\infty} \left(\int_{-\infty}^0 |t-s|^{-2} ds \right) |f_1(t)|^2 dt \\ = & \int_0^{+\infty} |f_1(t)|^2/t dt < +\infty \end{aligned}$$

Proof of surjectivity

Start with $f_0 \in H^{3/2}(\mathbb{R}_+)$, $f_1 \in H^{1/2}(\mathbb{R}_+)$ fulfilling (a, b_1, b_2)

$\tilde{f}_1 \in H^{1/2}(\mathbb{R})$ since:

$$
\begin{array}{llll}
 & \int_{\mathbb{R}^2} |\tilde{f}_1(t) - \tilde{f}_1(s)|^2/|t-s|^2 dtds & \overset{?}{<} +\infty & \\
= & \int_{\mathbb{R}^2_+} |f_1(t) - f_1(s)|^2/|t-s|^2 dtds & < +\infty & \text{by } H^{1/2}(\mathbb{R}_+) \\
+ & 2\int_0^{+\infty}\int_{-\infty}^0 |f_1(t)|^2/|t-s|^2 dtds & < +\infty & \text{by } b_1, b_2 \\
+ & \int_{\mathbb{R}^2_-} 0 dtds & =0 &
\end{array}
$$

The same for $\frac{\partial \tilde{f}_0}{\partial x}$ yields $\tilde{f}_0 \in H^{3/2}(\mathbb{R})$.

- By surjectivity in trace theorem for hyperplanes one has:
- $\exists w \in H^2(\mathbb{R} \times \mathbb{R}_+)$ with $w|_{y=0} = \tilde{f}_0$ and $\frac{\partial w}{\partial y}|_{y=0} = \tilde{f}_1$
- To complete the proof we need $w|_{x=0} = 0$.

Construction by mirror images

Define:

$$w'(x,y) = w(x,y) - c_1 w(-x,y) - c_2 w(-2x,y)$$

Now consider:

$$\begin{aligned} w'(0,y) &= w(0,y) - c_1 w(0,y) - c_2 w(0,y) \\ \frac{\partial w'}{\partial x}(0,y) &= \frac{\partial w}{\partial x}(0,y) + c_1 \frac{\partial w}{\partial x}(0,y) + 2c_2 \frac{\partial w'}{\partial x}(0,y) \end{aligned}$$

Thus:

$$\begin{aligned} w'(0,y) &= 0 \\ \frac{\partial w'}{\partial x}(0,y) &= 0 \end{aligned} \quad \text{iff} \quad \begin{aligned} c_1 + c_2 &= 1 \\ c_1 + 2c_2 &= -1 \end{aligned}$$

w' has the same traces! For $x > 0$:

$$w'|_{y=0}(x) = \tilde{f}_0(x) - c_1 \tilde{f}_0(-x) - c_2 \tilde{f}_0(-2x) = \tilde{f}_0(x)$$

Proof of special case completed

At first consider the subspace E of $H^2(\Omega)$ with $g_0 = g_1 = 0$. Then $u \in E$ is equivalent to $\tilde{u} \in H^2(\mathbb{R} \times \mathbb{R}_+)$, where $\tilde{u}$ is the continuation of u by zero for $x < 0$.

Special case

The mapping $u \to \{f_0, f_1\}$ has a unique continuous extension from E **onto** the subspace of $H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+)$ defined by:

- a $f_0(0) = 0$
- b_1 $\int_0^{+\infty} |\frac{\partial f_0}{\partial x}(t)|^2/t dt < +\infty$
- b_2 $\int_0^{+\infty} |f_1(t)|^2/t dt < +\infty$

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Lemma

For $u \in H^m(\Omega)$ f_0 and g_0 fulfill:

a $f_0(0) = g_0(0)$ if $m > 1$

b $\int_0^{+\infty} |f_0(t) - g_0(t)|^2/tdt < +\infty$ if m=1

Lemma

For $u \in H^m(\Omega)$ f_0 and g_0 fulfill:

a $f_0(0) = g_0(0)$ if $m > 1$

b $\int_0^{+\infty} |f_0(t) - g_0(t)|^2/tdt < +\infty$ if m=1

Proof: Condition a is obvious because u is continuous by Sobolev's Theorem then. Condition b holds because there is a constant C such that:

$$\int_0^{+\infty} |f_0(t) - g_0(t)|^2/tdt \leq C||u||_{1,\Omega}^2$$

For smooth u write:

$$\begin{aligned} f_0(t) - g_0(t) &= u(t,0) - u(t,t) + u(t,t) - u(0,t) \\ &= \int_0^t \frac{\partial u}{\partial x}(s,t) - \frac{\partial u}{\partial y}(t,s)ds \end{aligned}$$

Proof continues

Applying Cauchy-Schwarz equation

$$a(s,t) = \frac{\partial u}{\partial x}(s,t) - \frac{\partial u}{\partial y}(t,s)$$

$$\left(\int_0^t a(s,t)\cdot 1 ds\right)^2 \leq \int_0^t a(s,t)^2 ds \int_0^t 1^2 ds$$

$$\int_0^{+\infty} \frac{1}{t}\left(\int_0^t a(s,t)\cdot 1 ds\right)^2 dt \leq \int_0^{+\infty}\int_0^t a(s,t)^2 ds dt$$

Proof continues

Applying Cauchy-Schwarz equation

$$a(s,t) = \frac{\partial u}{\partial x}(s,t) - \frac{\partial u}{\partial y}(t,s)$$

$$\left(\int_0^t a(s,t)\cdot 1 ds\right)^2 \leq \int_0^t a(s,t)^2 ds \int_0^t 1^2 ds$$

$$\int_0^{+\infty} \frac{1}{t}\left(\int_0^t a(s,t)\cdot 1 ds\right)^2 dt \leq \int_0^{+\infty}\int_0^t a(s,t)^2 dsdt$$

and some geometry

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \leq 2a^2 + 2b^2 \\ 0 \leq (a-b)^2 &= a^2 - 2ab + b^2 \implies 2ab \leq a^2 + b^2 \end{aligned}$$

Proof complete

put together

$$\begin{aligned} & \int_0^{+\infty} \int_0^t \left(\frac{\partial u}{\partial x}(s,t) - \frac{\partial u}{\partial y}(t,s) \right)^2 dsdt \\ \leq \ & 2 \int_0^{+\infty} \int_0^t \left| \frac{\partial u}{\partial x}(s,t) \right|^2 + \left| \frac{\partial u}{\partial y}(t,s) \right|^2 dsdt \\ \leq \ & 2||u||_{1,\Omega}^2 \end{aligned}$$

Due to density the estimate remains valid for all $u \in H^1(\Omega)$ which completes the proof of condition b).

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Continuation

For now let Ω be the half-plane $x > 0$. Then any $u \in H^2(\Omega)$ can be extended into a function in $H^2(\mathbb{R}^2)$.

Proof by mirror images

For $x < 0$ define: $P_m u(x, y) = c_1 u(-x, y) + c_2 u(-2x, y)$

$$\begin{aligned} P_m u(0^-, y) &= c_1 u(0^+, y) + c_2 u(0^+, y) \\ \frac{\partial P_m u}{\partial x}|_{x=0^-} &= -c_1 \frac{\partial u}{\partial x}|_{x=0^+} - 2c_2 \frac{\partial u}{\partial x}|_{x=0^+} \end{aligned}$$

Thus $P_m u \in H^2(\mathbb{R}^2)$ iff $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 1$. This works for non-smooth u as well by density, as one easily shows: $||P_m u||_{2,\mathbb{R}^2} \leq C||u||_{2,\Omega}$

Continuation from a quarter of space to a half-space work as well, and the same procedure works in $\mathbb{R}^1$.

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Trace Theorem for quadrant

Let $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$. The mapping:

$$\begin{aligned} u &\rightarrow \{f_0, f_1, g_0, g_1\} \\ f_0 &= u|_{y=0}, f_1 = \frac{\partial u}{\partial y}|_{y=0} \\ g_0 &= u|_{x=0}, g_1 = \frac{\partial u}{\partial x}|_{x=0} \end{aligned}$$

defined for smooth u has a unique continuous extension from $H^2(\Omega)$ **onto** the subspace of

$$T = H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+) \times H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+)$$

defined by:

- a $f_0(0) = g_0(0)$
- b_1 $\int_0^{+\infty} |\frac{\partial f_0}{\partial x}(t) - g_1(t)|^2/tdt < +\infty$
- b_2 $\int_0^{+\infty} |f_1(t) - \frac{\partial g_0}{\partial y}(t)|^2/tdt < +\infty$

Proof necessity

- By the continuation property we can extend $u \in H^2(\Omega)$ into $U \in H^2(\mathbb{R}^2)$.
- For U the trace theorem on hyperplanes can be applied:

$$\{f_0, f_1, g_0, g_1\} \in T = H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+) \times H^{3/2}(\mathbb{R}_+) \times H^{1/2}(\mathbb{R}_+)$$

- Now applying the Lemma to to u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ proves the necessity of the conditions a, b_1, b_2.

Proof sufficiency 1

- Start from functions $\{f_0, f_1, g_0, g_1\} \in T$ fulfilling the conditions:
- Use continuation on $\{g_0, g_1\}$ to $\{G_0, G_1\} \in H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$
- By surjectivity of the trace theorem on hyperplanes there is a $V \in H^2(\mathbb{R}^2)$ with $\frac{\partial^k V}{\partial x^k}|_{x=0} = G_k,\ k = 0, 1$
- Now search for w such that $\frac{\partial^k w}{\partial y^k}|_{y=0} = f_k - \frac{\partial^k V}{\partial y^k}|_{y=0}$ and $\frac{\partial^k w}{\partial x^k}|_{x=0} = 0$ taking use of the special case.
- Then $u = V|_\Omega + w$ will have the required traces.

Proof sufficiency 2

- Define: $\phi_0 = V|_{y=0}$, $\phi_1 = \frac{\partial V}{\partial y}|_{y=0}$
- By the necessity part of the Theorem we have:

$$\begin{aligned} a* &\quad \phi_0(0) = g_0(0) \\ b_1* &\quad \int_0^{+\infty} |\frac{\partial \phi_0}{\partial x}(t) - g_1(t)|^2/tdt < +\infty \\ b_2* &\quad \int_0^{+\infty} |\phi_1(t) - \frac{\partial g_0}{\partial y}(t)|^2/tdt < +\infty \end{aligned}$$

- Then for $\psi_k = f_k - \phi_k$ one has:

$$\begin{aligned} a** &\quad \psi_0(0) = 0 \\ b_1** &\quad \int_0^{+\infty} |\psi_1(t)|^2/tdt < +\infty \\ b_2** &\quad \int_0^{+\infty} |\frac{\partial \psi_0}{\partial x}|^2/tdt < +\infty \end{aligned}$$

- Conditions $**$ follow from $a, b_{1,2}$ and $a*, b_{1,2}*$ with triangle inequality
- These are the assumptions of the special case, so the existence of w is proven.

Trace Theorem for quadrant

Let $\Omega = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$. The mapping:

$$\begin{aligned} u &\to \{f_0, f_1, g_0, g_1\} \\ f_0 &= u|_{y=0}, f_1 = \frac{\partial u}{\partial y}|_{y=0} \\ g_0 &= u|_{x=0}, g_1 = \frac{\partial u}{\partial x}|_{x=0} \end{aligned}$$

defined for smooth u has a unique continuous extension from $H^2(\Omega)$ **onto** the subspace of

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Sector of arbitrary angle

- For a sector with angle $\omega \in (0, \pi)$ just apply a linear change of coordinates
- Then apply the Theorem, because $||.||_{m,\Omega}$ remain unaffected
- The traces then become: $\{f_i = \gamma_1 \frac{\partial^i}{\partial \tau_2^i} u\}$,$\{g_i = \gamma_2 \frac{\partial^i}{\partial \tau_1^i} u\}$
- One prefers having traces in terms of: $\{F_i = \gamma_1 \frac{\partial^i}{\partial \nu_1^i} u\}$,$\{G_i = \gamma_2 \frac{\partial^i}{\partial \nu_2^i} u\}$
- $\tau_{1,2}$ and $\nu_{1,2}$ are tangential and normal vectors of the line segments
- we can have: $\begin{aligned} \tau_2 &= \alpha\tau_1 + \beta\nu_1 \\ \nu_2 &= \beta\tau_1 - \alpha\nu_1 \end{aligned}$

then we have new conditions: (in H^2)

$$\begin{aligned} g_1(0) &\equiv \frac{\partial f_0}{\partial t}(0) \\ \frac{\partial g_0}{\partial t}(0) &\equiv f_1(0) \end{aligned} \rightarrow \begin{aligned} G_1(0) &\equiv \beta\frac{\partial F_0}{\partial t}(0) - \alpha F_1(0) \\ \frac{\partial G_0}{\partial t}(0) &\equiv \alpha\frac{\partial F_0}{\partial t}(0) + \beta F_1(0) \end{aligned}$$

Non-convex Vertices

- Linear change of coordinats doesn't work for $\omega \geq \pi$
- For $\omega = \pi$ we have the trace theorem on hyperplanes
- with $\begin{aligned} A\vec{u} &= e_1 \\ A\vec{v} &= e_2 \end{aligned}$ we always have $A\frac{\vec{u}+\vec{v}}{2} = +\frac{e_1+e_2}{2}$
- Thus we need to consider the case Ω is a three-quarter-space
- The Theorem is valid for three-quarter-space as well, by the continuation property

$\Longrightarrow$ then we have the Theorem for arbitrary angels

Continuation from three-quarter space

Remember

$$P_m u = \begin{cases} u(x,y) & x > 0 \\ c_1 u(-x,y) + c_2 u(-2x,y) & x < 0 \end{cases}$$

first note

$$u(x, y > 0) = 0 \implies P_m u(x, y > 0) = 0$$

then one has

- $V = P_m(u|_{y>0})$
- $w = (u - V|_\Omega)|_{x>0}$
- $w(x, y > 0) = 0 \implies W = P_m w(x, y > 0) = 0$
- Now $U = V + W$ is the continuation
- for $y > 0$: $W = 0, U = V, V|_{y>0} = u$
- for $y < 0$ and $x > 0$: $U = V + W = V + (u - V|_\Omega) = u$

Connection to finite domains

The step from a single corner to a finite domain with multiple vertices is done by partition of unity

- This is a method used regulary in Sobolev Space to localise certain properties
- I only give a sketch of the idea here:
- In our case we have one function per corner $c_i(x, y)$ with values in $[0, 1]$
- $\forall (x, y) \in \Omega : \sum_{i=1}^{N} c_i(x, y) = 1$
- close to corner i the function $c_i = 1$ all others are zero
- we have properties for every single plane sector G_i from Theorem
- One can "glue" it together

End

Thank you for your attention!